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Mathematics of Countour Classes

Ilhan M. Izmirli[1]

Abstract

Regardless of their degree of musical sophistication and their cultural differences, all who listen to music have an innate feel for proximity (distance between consecutive notes, or what note is expected to follow the other), and direction (whether the melody is ascending or descending). In [7], I had looked into the mathematics of pitch spaces where proximity was the distinguishing characteristic. In this paper, I will investigate the properties of pitch spaces where only the direction counts, that is, spaces where we are not interested in the distance between the pitches, but only in whether a pitch is higher than, lower than, or the same as another pitch. The resulting constructs, the so called **contour classes**, are used as sequences in fugues, as leitmotifs in operas, and as changes of mode from major to minor in variations.

1. Basic Definitions

There are numerous papers and books ([1], [2], [4], [5], [6], [7], [8], [9], [10], [11]) that emphasize the connection between mathematics and music based on (and mostly confined to) theories of harmony and temperaments, and more than a few that deal with the specific aspects of twentieth century music ([3], [12], [13], and [14]) which is, in fact, quite mathematics-based.

Let us, here, review the basic terminology that will be used throughout this paper. Since music is an art form that deals with the permutation of tones in time, we start with the definition of a tone. A ***tone*** is defined as a sound that has a definite ***pitch*** (frequency), ***duration***, ***timbre*** (tone color), and ***dynamics*** (loudness).

A ***pitch space*** is any collection of tones. For example, if we let $FQ(s)$denote the fundamental frequency of a pitch s, we can define a pitch space

$$S = \{2^a 3^b 5^c FQ(s) \mid s \text{ is any fixed pitch, and } a, b, c \in Z\}$$

This space is called ***space of just intonation***. We can also talk about the pitch space

$$S = \{\text{ C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B, …}\}$$

called the ***$\boldsymbol{p}$-space*** (***the space of chromatic pitches***). A closely related pitch space is the ***$\boldsymbol{u}$-space*** (***the space of diatonic pitches***)

S ={C, D, E, F, G, A, B, C, ...}

In this paper, we will work with a different type of space, called the ***contour space***:

[1]George Mason University. Email: iizmirl2@gmu.edu

Definition 1.1 A ***contour space*** (**c-space**) ***of order*** $\boldsymbol{n}$ is a pitch space of n elements called ***c-pitches*** (*cp*s), numbered in order from low to high, beginning with 0.

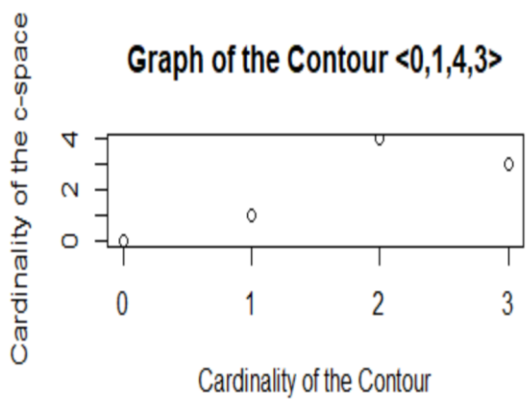
Obviously, there are 2^n subsets of a c-space of order n. Each such subset is called a ***cpset***. One can establish an equivalence relation by letting cp sets that have the same cardinality to belong to the same equivalence class. Note that in a c-space of order n, the class of sets with cardinality k, with $0 \leq k \leq n$, will have $\binom{n}{k}$ members.

The most important subsets of a c-space are the ***ordered subsets***.

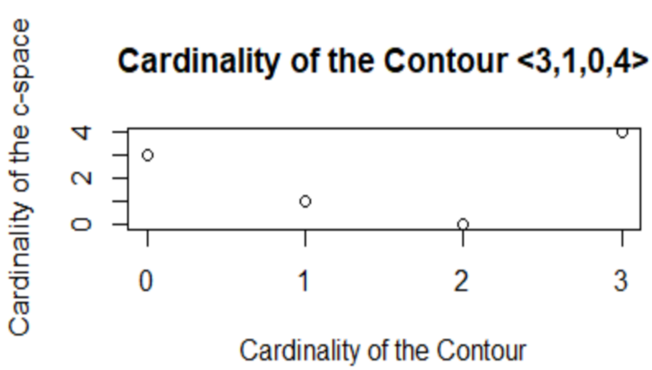
Definition 1.2 Any ordered subsets of a c-space is called a ***segment*** or a ***contour***, and is denoted as$\langle x\ y\ \ z\,..\rangle$.

Thus, although the cp sets $\{0, 1, 3\}$ and $\{1, 3, 0\}$ are equivalent, the contours $\langle 0\ 1\ 3\rangle$ and $\langle 1\ \ 3\ \ 0\rangle$ are different. Contours are denoted by capital letters, Q, ... The notation Q_j stands for the$(j + 1)$st element from the left of contour$Q$.

A useful depiction of a contour is a graphical one, a scatter plot, which is really a cross between a mathematical graph and musical notation. On the horizontal axis, we put $0, 1, \ldots, k - 1$, where k is the cardinality of the contour, and on the vertical axis we put $0, 1, \ldots, n - 1$, where n is the order of the c-space and plot the points. For example, let us assume we have a c-space of order 5, and we want to graph contour $\langle 0\ 1\ 4\ \ 3\rangle$

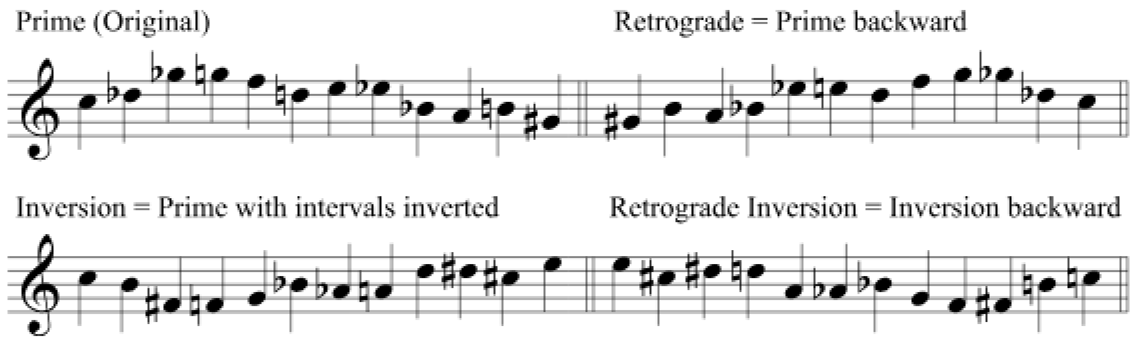


On the other hand, the contour $\langle 3\ \ 1\ \ 0\ \ 4\rangle$ will have the graph



Definition 1.3 Let a sequence of pitches be called the ***prime*** and be denoted as P.Then, the ***retrograde*** R is defined as the prime in reverse order, the ***inversion***I as the prime with inverted intervals, and the ***retrograde inversion*** RI as the retrograde of the inversion of the prime. The prime sequence itself is usually denoted by P.

So, for example,



Definition 1.4 Let p, qbe two c-pitches in a c-space. We define a function $C(p, q)$, called a ***comparison function*** as

$$C(p,q) = \begin{cases} 1 & \text{if } q \text{ is higher than } p \\ 0 & \text{if } q \text{ is the same as } p \\ -1 & \text{if } q \text{ is lower than } p \end{cases}$$

Thus, $C(1,3) = 1, C(2,2) = 0,$ and $C(4,2) = -1$.

The following lemma follows immediately from Definition 1.4:

Lemma 1.1 $C(p,q) = -C(q,p)$

Given any contour, we would like to have the entire comparisons of all the pitches in that contour. This is given by the following ***comparison matrix*** where the segment is written as a row and a column, and $C(p,q)$ is calculated for each p in the row and each q in the column.Obviously,

- The comparison matrix has symmetry of inverse signs (by Lemma 1.1)
- Its diagonal elements are 0 (by Definition 1.4).

For instance, for the contour $\langle 2\ \ 1\ \ 4\ \ 5 \rangle$, we have the comparison matrix

$$2 \quad 1 \quad 4 \quad 5$$

$$20 - 111$$

$$110\ 1 \quad 1$$

$$4 - 1 - 101$$

$$5 - 1\ - 1\ \ - 1$$

Clearly, there are several contours that give rise to the same comparison matrix. For example, the contour $\langle 2\ \ 1\ \ 4\ \ 5 \rangle$ given above, and the contour $\langle 6\ \ 4\ \ 7\ \ 9 \rangle$ are easily seen to have this property.

Definition 1.5 Two contours are said to be ***equivalent*** (or ***contour equivalent***) if and only if they generate the same comparison matrix. Contour equivalence is sometimes called ***contour preservation***.

Using this equivalence relation, one might partition contours into ***contour classes***. It can easily be shown that equivalent contours will have the same graphical pattern.

Let us look at some examples.

(i) In a c-space of order one, there is one contour class, $\langle 0 \rangle$.

(ii) In a c-space of order two, that is, the c-space consisting of {0 , 1}, we have three contour classes; $\langle 0 \rangle$, $\langle 0\ 1 \rangle$ and $\langle 1\ \ 0 \rangle$.

(iii) Now in case of c-spaces of order three, namely, the space {0 , 1 , 2} since the comparison matrices for contours $\langle 0\ \ 1 \rangle$, $\langle 0\ \ 2 \rangle$, and $\langle 1\ \ 2 \rangle$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & -1 & 0 \end{array} \qquad \begin{array}{c|cc} & 0 & 2 \\ \hline 0 & 0 & 1 \\ 2 & -1 & 0 \end{array} \qquad \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & -1 & 0 \end{array}$$

are the same, these contours are equivalent, and we take $\langle 0\ \ 1 \rangle$as their representative. Similarly, the comparison matrices for the contours $\langle 1\ \ 0 \rangle$, $\langle 2\ \ 0 \rangle$, $\langle 2\ \ 1 \rangle$ are the same, and we take $\langle 1\ \ 0 \rangle$as the representative of these equivalent contours. Direct construction shows that each one of the contours

$$\langle 0\ \ 1\ \ 2 \rangle, \langle 0\ \ 2\ \ 1 \rangle, \langle 1\ \ 0\ \ 2 \rangle, \langle 1\ \ 2\ \ 0 \rangle, \langle 2\ \ 0\ \ 1 \rangle, \text{ and } \langle 2\ \ 1\ \ 0 \rangle$$

give rise to different comparison matrices. Thus, for a c-space of order three, we have the following 9 contour classes:

$$\langle 0 \rangle, \langle 0\ \ 1 \rangle, \langle 1\ \ 0 \rangle, \langle 0\ \ 1\ \ 2 \rangle, \langle 0\ \ 2\ \ 1 \rangle, \langle 1\ \ 0\ \ 2 \rangle, \langle 1\ \ 2\ \ 0 \rangle, \langle 2\ \ 0\ \ 1 \rangle, \text{ and } \langle 2\ \ 1\ \ 0 \rangle$$

(iv) Proceeding likewise, one can easily see that the contour classes of a c-space of order 4 to be

$$\langle 0 \rangle, \langle 0\ \ 1 \rangle, \langle 1\ \ 0 \rangle, \langle 0\ \ 1\ \ 2 \rangle, \langle 0\ \ 2\ \ 1 \rangle, \langle 1\ \ 0\ \ 2 \rangle, \langle 1\ \ 2\ \ 0 \rangle, \langle 2\ \ 0\ \ 1 \rangle, \langle 2\ \ 1\ \ 0 \rangle, \langle 0\ 1\ \ 2\ \ 3 \rangle,$$
$$\langle 0\ \ 1\ \ 3\ \ 2 \rangle, \langle 0\ \ 2\ \ 1\ \ 3 \rangle, \langle 0\ \ 2\ \ 3\ \ 1 \rangle, \langle 0\ \ 3\ \ 1\ 2 \rangle, \langle 0\ \ 3\ \ 2\ \ 1 \rangle, \langle 1\ \ 0\ \ 2\ \ 3 \rangle, \langle 1\ \ 0\ \ 3\ \ 2 \rangle,$$
$$\langle 1\ \ 2\ \ 0\ \ 3 \rangle, \langle 1\ \ 2\ \ 3\ \ 0 \rangle, \langle 1\ \ 3\ \ 0\ \ 2\ \rangle, \langle 1\ \ 3\ \ 2\ \ 0\ \rangle, \langle 2\ \ 0\ \ 1\ \ 3 \rangle, \langle 2\ \ 0\ \ 3\ \ 1 \rangle, \langle 2\ \ 1\ \ 0\ \ 3 \rangle,$$
$$\langle 2\ \ 1\ \ 3\ \ 0 \rangle, \langle 2\ \ 3\ \ 0\ \ 1 \rangle, \langle 2\ \ 3\ \ 1\ \ 0 \rangle, \langle 3\ \ 0\ \ 1\ \ 2 \rangle, \langle 3\ \ 0\ \ 2\ \ 1 \rangle, \langle 3\ \ 1\ \ 0\ \ 2 \rangle, \langle 3\ \ 1\ \ 2\ \ 0 \rangle,$$
$$\langle 3\ \ 2\ \ 0\ \ 1 \rangle, \langle 3\ \ 2\ \ 1\ \ 0 \rangle$$

i.e., we have thirty-three contour classes.

It is easy to see inductively that as we pass from a c-space of order $m-1$ to a one of order m, we increase the number of contour classes by $m!$. This proves

Theorem 1.1 Number of contour classes in a c-space of order n is $n! + n - 1$.

2. Operations on Contours

In this section we will explore some properties of the operators I, R, and IR on c-spaces.

Definition 2.1 Let Q be a contour in a c-space of order n. Then, for any $j = 0.1, \dots, n-1$,

$$IQ_j = (n-1) - Q_j$$

So if we have a contour space of order six, the inverse of the contour $Q = \langle 0\ \ 2\ \ 4\ \ 5 \rangle$will be the contour $IQ = \langle 5\ \ 3\ \ 1\ \ 0 \rangle$

Lemma 2.1. Let Q be a contour in a c-space of order n. Then

$$I(I(Q) = Q$$

Proof. For each $j = 0.1, \dots, n-1, I(I(Q_j)) = (n-1) - \{(n-1) - Q_j\} = Q_j$

Theorem 2.1 Let Q be a contour in a c-space of order n. Then

$$C(IQ_j, IQ_k) = -C(Q_j, Q_k)$$

for all j, k $0.1, \dots, n-1$.

Proof. If $j = k$, then the above equality holds since both sides are equal to zero. If $Q_j < Q_k$, then

$$IQ_j = (n-1) - Q_j > (n-1) - Q_k = IQ_k$$

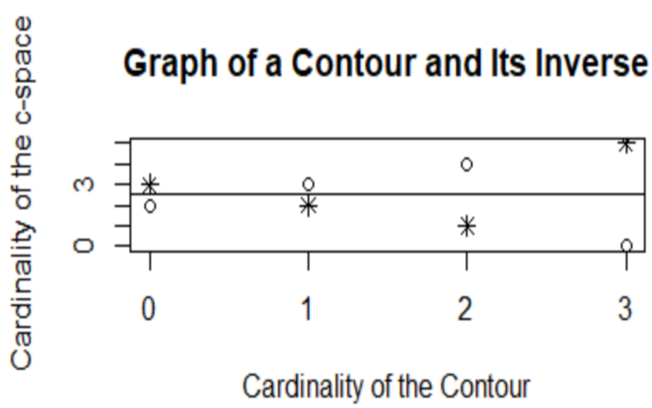
Corollary 2.1 Let Q be a contour in a c-space of order n. Let A be comparison matrix for Q, and B the comparison matrix for IQ. Then

a) $A = B^T$
b) $A + B = 0$

Proof. Follows directly from Theorem 2.1.

For example, in a c-space of order six, if $Q = \langle 2\ 3\ 4\ 0\rangle$, its inverse IQwill be $\langle 3\ 2\ 1\ 5\rangle$, and Theorem 2.1 and Corollary 2.1 can easily be verified.

If we plot graphs of these contours, we notice an interesting property: graphs are symmetric with respect to the horizontal line drawn at $\frac{n-1}{2}$. In the graph below, o's denote the elements of Q, and *'s denote the elements of IQ.



Definition 2.2 Let Q be contour of cardinality k in a c-space of order n. Then, its retrograde by RQ is defined as

$$RQ_j = Q_{k-1-j}$$

for $j = 0, 1, \dots, k$.

Note that the operators $R$and I commute.

Lemma 2.2 Let Q be a contour in a c-space of order n. Then, $R(R(Q)) = Q$

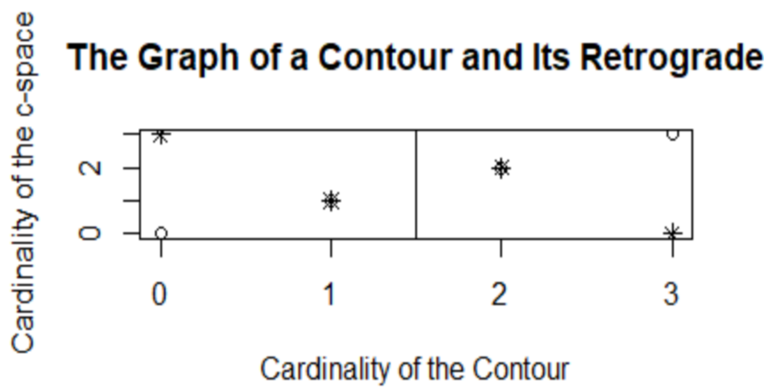
Proof. This follows since for each $j = 0, 1, \dots, k$, $(k-1) - \{(k-1) - j\} = j$

For any matrix A, let us let $A^{T'}$ denote the matrix A transposed with respect to its second diagonal. Then we have the following

Corollary 2.2 Let Q be a contour in a c-space of order n, and let A be comparison matrix for Q. Then the comparison matrix for RQ is $-A^{T'}$

Proof. Follows directly from Definition 2.2.

Let Q be a contour in a c-space of order n with cardinality k. We note the following interesting property: the graphs of Q and RQ are symmetric with respect to the vertical line drawn at $\frac{k-1}{2}$. For example, if $Q = \langle 0\ 2\ 1\ 3 \rangle$, then $RQ = \langle 3\ 1\ 2\ 0 \rangle$, and if we show elements of contour Q by x's, and elements of contour RQ by o's,



Lemma 2.3 Let Q be a contour in a c-space of order n. Then

$$C(RQ_j, RQ_k) = -C(Q_j, Q_k)$$

Proof. Follows directly from Definition 2.2.

The following theorem follows easily from Theorem 2.1 and Lemma 2.3:

Theorem 2.2 Let Q be a contour in a c-space of order n. Then, for all $j, k\ 0.1, \ldots, n-1$.

$$C(IRQ_j, IRQ_k) = C(Q_j, Q_k)$$

Now, Theorem 2.2 implies that some contours will be IR invariant.

Theorem 2.3 Let Q be a contour of cardinality k in a c-space of order n. Then, Q will be IR invariant if for all $j = 0, 1, .., k$,

$$Q_j + Q_{k-1-j} = n - 1$$

Proof. By Definition 2.2,

$$RQ_j = Q_{k-1-j}$$

and by Definition 2.1

$$IRQ_j = n - 1 - Q_{k-1-j}$$

Since IRinvariance means that $IRQ_j = Q_j$, the result follows.

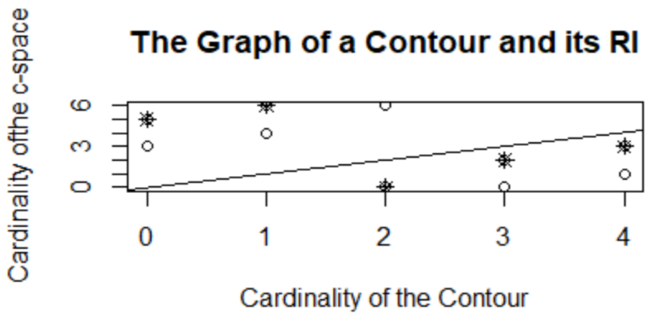
So for example, if n = 6, the contour$Q = \langle 1\ 2\ 3\ 4 \rangle$will be RI invariant. A contour class is RIinvariant if all its members are RI invariant.

Definition 2.3 Two contours are R-*related* (or similarly, IR-*related*) if one is the retrograde (or retrograde inverse) of the other.

The following theorem is a direct consequence of Definition 2.3:

Theorem 2.4 Let Q be a contour in a c-space of order n, and let Q and Q' be RI-related. Let the comparison matrix of Q be A. Then the comparison matrix of Q' is $A^{T'}$

The graphs of such contours will be symmetric with respect to the diagonal. For example, if we observe the graphs of two RI-related contours $\langle 3\ \ 4\ \ 6\ \ 0\ \ 1\rangle$ and $\langle 5\ \ 6\ \ 0\ \ 2\ \ 3\rangle$



A contour class is ***R-related*** (or ***IR-related***) if every member of one is the retrograde (or retrograde inverse) of a member of the other.

Obviously, in any contour space, some of the non-replicative contours can be obtained from others by the operations I, R, and IR. This partitioning of contours is called ***segment classes***. Now, of all the 33 contour classes

<0>, <0 1>, <1 0>, <0 1 2>, <0 2 1>, <1 0 2>, <1 2 0>, <2 0 1>, <2 1 0>, <0 1 2 3>,
<0 1 3 2>, <0 2 1 3>, <0 2 3 1>, <0 3 1 2>, <0 3 2 1>, <1 0 2 3>, <1 0 3 2>,
<1 2 0 3>, < 1 2 3 0>, <1 3 0 2>, <1 3 2 0>, <2 0 1 3>, <2 0 3 1>, <2 1 0 3>,
<2 1 3 0>, <2 3 0 1>, <2 3 1 0>, <3 0 1 2>, <3 0 2 1>, <3 1 0 2>, <3 1 2 0>,
<3 2 0 1>, <3 2 1 0>

in a contour class of order 4, we have the following 12 segment classes:

<0>, <0 1>, <0 1 2>, <0 2 1>, <0 1 2 3>, <0 1 3 2>, <0 2 1 3>, <0 2 3 1>, <0 3 1 2>, <0 3 2 1>,
<1 0 3 2>, <1 3 0 2>

3. Group Theory and c-spaces

There is also an interesting connection between group theory and c-spaces.

Let us consider, for a given integer n the set of symbols

$$D = \left\{x^j y^k \,|\, j = 0, 1; k = 0, 1, \ldots, n-1\right\}$$

with the following rules:

1. $x^j y^k = x^{j'} y^{k'}$ if and only if $i = i'$ and $j = j'$
2. $x^2 = e$
3. $y^n = e$
4. $xy = y^{-1}x$

where$e$ denotes $x^0 y^0$ and y^{-1} is defined by the formula

$$yy^{-1} = y^{-1}y = e$$

It is easy to show that D is a non-abelian group of order $2n$. It is called the ***dihedral group.*** By interpreting y as a rotation of the Euclidean plane about the origin through an angle of $\frac{2\pi}{n}$, and x as a reflection about the vertical axis, the dihedral group can be interpreted as the group of rigid motions that leave a regular n-gon invariant.

An interesting special case is the dihedral group of order 4 (rigid motions that leave a square invariant). Here x is reflection about the vertical axis, and y is a rotation through $90°$. The group elements are e, x, y, and xy, and the group table is

	e	x	y	xy
e	e	x	y	xy
x	x	e	xy	y
y	y	xy	e	x
xy	xy	y	x	e

This is an abelian group called the ***Klein four group***.

Theorem 3.1*The four operator*s $P, I, R,$ *and*RI*form a Klein four group.*

Proof. Result follows by replacing$e$ by P, x by R, y by I, and xy by RI.

The following corollary is obvious:

Corollary 3.1 The subgroups of the above group are$\{P\}, \{P, R\}, \{P, RI\}$, and $\{P, I\}$

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