

## Structure of essential spectra and discrete spectrum of the energy operator of five electron systems in the Hubbard Model. Sextet and quartet states

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### Abstract

We consider a five-electron system in the Hubbard model with a coupling between nearest-neighbors. The structure of essential spectrum and discrete spectrum of the systems in the sextet, and first, second, and third quartet states in a  $v$ -dimensional lattice are investigated.

**Keywords:** essential spectrum, discrete spectrum, five-electron system, bound state, anti-bound state, Hubbard model, doublet state, sextet state, quartet state.

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### 1. Introduction

In the early 1970s, three papers [1]-[3], where a simple model of a metal was proposed that has become a fundamental model in the theory of strongly correlated electron systems, appeared almost simultaneously and independently. In that model, a single nondegenerate electron band with a local Coulomb interaction is considered. The model Hamiltonian contains only two parameters: the matrix element  $t$  of electron hopping from a lattice site to a neighboring site and the parameter  $U$  of the on-site Coulomb repulsion of two electrons. In the secondary quantization representation, the Hamiltonian can be written as

$$H = t \sum_{m,\gamma} a_{m,\gamma}^+ a_{m,\gamma} + U \sum_m a_{m,\uparrow}^+ a_{m,\uparrow} a_{m,\downarrow}^+ a_{m,\downarrow}, \quad (1)$$

where  $a_{m,\gamma}^+$  and  $a_{m,\gamma}$  denote Fermi operators of creation and annihilation of an electron with spin  $\gamma$  on a site  $m$  and the summation over  $\tau$  means summation over the nearest neighbors on the lattice.

The model proposed in [1]-[3] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system, although the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson [4]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model [5], which had appeared 30 years before [1]-[3]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account. The Hubbard model is an approximation used in solid state physics to describe the transition between conducting and insulating states. It is the simplest model describing particle interaction on a lattice. Its Hamiltonian contains only two terms: the kinetic term corresponding to the tunneling (hopping) of particles between lattice sites and a term corresponding to the on-site interaction. Particles can be fermions, as in Hubbard's original work, and also bosons. The simplicity and sufficiency of Hamiltonian (1) have made the Hubbard model very popular and effective for describing strongly correlated electron systems.

The Hubbard model well describes the behavior of particles in a periodic potential at sufficiently low temperatures such that all particles are in the lower Bloch band and long-range interactions can be neglected. If the interaction between particles on different sites is taken into account, then the model is often called the extended Hubbard model. It was proposed for describing electrons in solids, and it remains especially interesting since then for studying high-temperature superconductivity. Later, the extended Hubbard model also found applications in describing the behavior of ultracold atoms in optical lattices. In considering electrons in solids, the Hubbard model can be considered a sophisticated version of the model of strongly bound electrons, involving only the electron hopping term in the Hamiltonian. In the case of strong interactions, these two models can give essentially different results. The Hubbard model exactly predicts the existence of so-called Mott insulators, where conductance is absent due to strong repulsion between particles. The

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Hubbard model is based on the approximation of strongly coupled electrons. In the strongcoupling approximation, electrons initially occupy orbitals in atoms (lattice sites) and then hop over to other atoms, thus conducting the current. Mathematically, this is represented by the so-called hopping integral. This process can be considered the physical phenomenon underlying the occurrence of electron bands in crystal materials. But the interaction between electrons is not considered in more general band theories. In addition to the hopping integral, which explains the conductance of the material, the Hubbard model contains the so-called on-site repulsion, corresponding to the Coulomb repulsion between electrons. This leads to a competition between the hopping integral, which depends on the mutual position of lattice sites, and the on-site repulsion, which is independent of the atom positions. As a result, the Hubbard model explains the metal–insulator transition in oxides of some transition metals. When such a material is heated, the distance between nearest-neighbor sites increases, the hopping integral decreases, and on-site repulsion becomes dominant.

The Hubbard model is currently one of the most extensively studied multielectron models of metals [6]-[10]. Therefore, obtaining exact results for the spectrum and wave functions of the crystal described by the Hubbard model is of great interest. The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [6]. It is known that two-electron systems can be in two states, triplet and singlet [6]-[10]. It was proved in [6] that the spectrum of the system Hamiltonian  $H^t$  in the triplet state is purely continuous and coincides with a segment  $[m, M] = [2A - 4Bv, 2A + 4Bv]$ , and the operator  $H^s$  of the system in the singlet state, in addition to the continuous spectrum  $[m, M]$ , has a unique antibound state for some values of the quasimomentum. For the antibound state, correlated motion of the electrons is realized under which the contribution of binary states is large. Because the system is closed, the energy must remain constant and large. This prevents the electrons from being separated by long distances. Next, an essential point is that bound states (sometimes called scattering-type states) do not form below the continuous spectrum. This can be easily understood because the interaction is repulsive. We note that a converse situation is realized for  $U < 0$ : below the continuous spectrum, there is a bound state (antibound states are absent) because the electrons are then attracted to one another. For the first band, the spectrum is independent of the parameter  $U$  of the on-site Coulomb interaction of two electrons and corresponds to the energy of two noninteracting electrons, being exactly equal to the triplet band. The second band is determined by Coulomb interaction to a much greater degree: both the amplitudes and the energy of two electrons depend on  $U$ , and the band itself disappears as  $U \rightarrow 0$  and increases without bound as  $U \rightarrow \infty$ . The second band largely corresponds to a one-particle state, namely, the motion of the doublet, i.e., two-electron bound states.

The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [11]. In the three-electron systems are exists quartet state, and two type doublet states. The quartet state corresponds to the free motion of three electrons over the lattice with the basic functions  $q_{m,n,p}^{3/2} = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{p,\uparrow}^+ \varphi_0$ . In the work [11] is proved that the essential spectrum of the system in a quartet state consists of a single segment and the three-electron bound state or the three-electron antibound state is absent. The doublet state corresponds to the basic functions  ${}^1d_{m,n,p}^{1/2} = a_{m,\uparrow}^+ a_{n,\downarrow}^+ a_{p,\uparrow}^+ \varphi_0$ , and  ${}^2d_{m,n,p}^{1/2} = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{p,\downarrow}^+ \varphi_0$ . If  $v = 1$  and  $U > 0$ , then the essential spectrum of the system of first doublet state operator  $\tilde{H}_1^d$  is exactly the union of three segments and the discrete spectrum of  $\tilde{H}_1^d$  consists of a single point, i.e., in the system exists unique antibound state. In the two-dimensional case, we have the analogous results. In the three-dimensional case, or the essential spectrum of the system in the first doublet state operator  $\tilde{H}_1^d$  is the union of three segments and the discrete spectrum of operator  $\tilde{H}_1^d$  consists of a single point, i.e., in the system exists only one antibound state, or the essential spectrum of the system in the first doublet state operator  $\tilde{H}_1^d$  is the union of two segments and the discrete spectrum of the operator  $\tilde{H}_1^d$  is empty, or the essential spectrum of the system in the first doublet state operator  $\tilde{H}_1^d$  is consists of a single segment, and discrete spectrum is empty, i.e., in the system the antibound state is absent. In the one-dimensional case, the essential spectrum of the operator  $\tilde{H}_2^d$  of second doublet state is the union of three segments, and the discrete spectrum of operator  $\tilde{H}_2^d$  consists of no more than one point. In the two-dimensional case, we have analogous results. In the three-dimensional case, or the essential spectrum of the system in the second doublet state operator  $\tilde{H}_2^d$  is the union of three segments and the discrete spectrum of operator  $\tilde{H}_2^d$  consists of no more than one point, i.e., in the system exists no more than one antibound state, or the essential spectrum of the system in the second doublet state operator  $\tilde{H}_2^d$  is the union of two segments and the discrete spectrum of the operator  $\tilde{H}_2^d$  is empty, or the essential spectrum of the system in the second doublet state operator  $\tilde{H}_2^d$  is consists of a single segment, and discrete spectrum is empty, i.e., in the system the antibound state is absent.

The spectrum of the energy operator of system of four electrons in a crystal described by the Hubbard Hamiltonian in the triplet state were studied in [12]. In the four-electron systems are exists quintet state, and three type triplet states, and two type singlet states. The triplet state corresponds to the basic functions  ${}^1t_{m,n,p,r}^1 = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{p,\uparrow}^+ a_{r,\downarrow}^+ \varphi_0$ ,  ${}^2t_{m,n,p,r}^1 = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{p,\downarrow}^+ a_{r,\uparrow}^+ \varphi_0$ ,  ${}^3t_{m,n,p,r}^1 = a_{m,\uparrow}^+ a_{n,\downarrow}^+ a_{p,\uparrow}^+ a_{r,\uparrow}^+ \varphi_0$ . If  $\nu=1$  and  $U > 0$ , then the essential spectrum of the system first triplet state operator  ${}^1\tilde{H}_t^1$  is exactly the union of two segments and the discrete spectrum of operator  ${}^1\tilde{H}_t^1$  is empty. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system first triplet-state operator  ${}^1\tilde{H}_t^1$  is the union of two segment and the discrete spectrum of operator  ${}^1\tilde{H}_t^1$  is empty, or the essential spectrum of the system first triplet-state operator  ${}^1\tilde{H}_t^1$  is single segment and the discrete spectrum of operator  ${}^1\tilde{H}_t^1$  is empty. If  $\nu=1$  and  $U > 0$ , then the essential spectrum of the system second triplet state operator  ${}^2\tilde{H}_t^1$  is exactly the union of three segments and the discrete spectrum of operator  ${}^2\tilde{H}_t^1$  is consists no more than one point. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system second triplet-state operator  ${}^2\tilde{H}_t^1$  is the union of three segments and the discrete spectrum of the operator  ${}^2\tilde{H}_t^1$  is consists no more than one point, or the essential spectrum of the system second triplet-state operator  ${}^2\tilde{H}_t^1$  is the union of two segments and the discrete spectrum of the system second triplet state operator  ${}^2\tilde{H}_t^1$  is empty, or the essential spectrum of the system second triplet-state operator  ${}^2\tilde{H}_t^1$  is consists of single segment and the discrete spectrum of the operator  ${}^2\tilde{H}_t^1$  is empty. If  $\nu = 1$  and  $U > 0$ , the essential spectrum of the system third triplet-state operator  ${}^3\tilde{H}_t^1$  is exactly the union of three segments and the discrete spectrum of the operator  ${}^3\tilde{H}_t^1$  is consists no more than one point. In two-dimensional case, we have analogous results. In the three-dimensional case, the essential spectrum of the system third triplet-state operator  ${}^3\tilde{H}_t^1$  is the union of three segments, and the discrete spectrum of the operator  ${}^3\tilde{H}_t^1$  is consists no more than one point or the essential spectrum of the system third triplet-state operator  ${}^3\tilde{H}_t^1$  is the union of two segments, and the discrete spectrum of the operator  ${}^3\tilde{H}_t^1$  is empty, or the essential spectrum of the system third triplet-state operator  ${}^3\tilde{H}_t^1$  is consists of single segment, and the discrete spectrum of the operator  ${}^3\tilde{H}_t^1$  is empty. We see that there are three triplet states, and they have different origins.

The spectrum of the energy operator of four-electron systems in the Hubbard model in the quintet, and singlet states were studied in [13]. The quintet state corresponds to the free motion of four electrons over the lattice with the basic functions  $q_{m,n,p,r}^2 = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{p,\uparrow}^+ a_{r,\uparrow}^+ \varphi_0$ . In the work [13] proved, that the spectrum of the system in a quintet state is purely continuous and coincides with the segment  $[4A - 8B\nu, 4A + 8B\nu]$ , and the four-electron bound states or the four-electron antibound states is absent. The singlet state corresponds to the basic functions  ${}^1s_{p,q,r,t}^0 = a_{p,\uparrow}^+ a_{q,\uparrow}^+ a_{r,\downarrow}^+ a_{t,\downarrow}^+ \varphi_0$ ,  ${}^2s_{p,q,r,t}^0 = a_{p,\uparrow}^+ a_{q,\downarrow}^+ a_{r,\uparrow}^+ a_{t,\downarrow}^+ \varphi_0$ , and these two singlet states have different origins.

If  $\nu = 1$  and  $U > 0$ , then the essential spectrum of the system of first singlet-state operator  ${}^1\tilde{H}_4^s$  is exactly the union of three segments and the discrete spectrum of the operator  ${}^1\tilde{H}_4^s$  is consists only one point. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system first singlet-state operator  ${}^1\tilde{H}_4^s$  is the union of three segments and the discrete spectrum of the operator  ${}^1\tilde{H}_4^s$  is consists only one point, or the essential spectrum of the system of first singlet-state operator  ${}^1\tilde{H}_4^s$  is the union of two segment and the discrete spectrum of the operator  ${}^1\tilde{H}_4^s$  is empty, or the essential spectrum of the system of first singlet-state operator  ${}^1\tilde{H}_4^s$  is consists of single segment and the discrete spectrum of operator  ${}^1\tilde{H}_4^s$  is empty.

If  $\nu = 1$  and  $U > 0$ , then the essential spectrum of the system of second singlet-state operator  ${}^2\tilde{H}_4^s$  is exactly the union of three segments and the discrete spectrum of operator  ${}^2\tilde{H}_4^s$  is consists only one point. In two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system second singlet-state operator  ${}^2\tilde{H}_4^s$  is the union of three segments and the discrete spectrum of the operator  ${}^2\tilde{H}_4^s$  is consists only one point, or the essential spectrum of the system of second singlet-state operator  ${}^2\tilde{H}_4^s$  is the union of two segment and the discrete spectrum of the operator  ${}^2\tilde{H}_4^s$  is empty, or the essential spectrum of the system of second singlet-state operator  ${}^2\tilde{H}_4^s$  is consists of single segment and the discrete spectrum of operator  ${}^2\tilde{H}_4^s$  is empty.

Here, we consider the energy operator of five-electron systems in the Hubbard model and describe the structure of the essential spectra and discrete spectrum of the system for sextet and first, second, third quartet states.

The Hamiltonian of the chosen model has the form

$$H = A \sum_{m,\gamma} a_{m,\gamma}^+ a_{m,\gamma} + B \sum_{m,\gamma} a_{m,\gamma}^+ a_{m+\tau,\gamma} + U \sum_m a_{m,\uparrow}^+ a_{m,\uparrow} a_{m,\downarrow}^+ a_{m,\downarrow}, \quad (2)$$

Here  $H$  is the electron energy at a lattice site,  $B$  is the transfer integral between neighboring sites (we assume that  $B > 0$  for convenience),  $\tau = \pm e_j$ ,  $j = 1, 2, \dots, \nu$ , where  $e_j$  are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors,  $U$  is the parameter of the on-site Coulomb interaction of two electrons,  $\gamma$  is the spin index,  $\gamma = \uparrow$  or  $\gamma = \downarrow$ ,  $\uparrow$  and  $\downarrow$  denote the spin values  $\frac{1}{2}$  and  $-\frac{1}{2}$ , and  $a_{m,\gamma}^+$  and  $a_{m,\gamma}$  are the respective electron creation and annihilation operators at a site  $m \in Z^\nu$ .

In the five-electron systems exists sextet state, four type quartet states, and five type doublet states. The energy of the system depends on its total spin  $S$ . Along with the Hamiltonian, the  $N_e$  electron system is characterized by the total spin  $S$ ,  $S = S_{max}, S_{max} - 1, \dots, S_{min}, S_{max} = \frac{N_e}{2}, S_{min} = 0, \frac{1}{2}$ . Hamiltonian (2) commutes with all components of the total spin operator  $S = (S^+, S^-, S^z)$ , and the structure of eigenfunctions and eigenvalues of the system therefore depends on  $S$ . The Hamiltonian  $H$  acts in the antisymmetric Fo'ck space  $\tilde{\mathcal{H}}_{as}$ . Below we give the constructions of the Fo'ck space  $F(\tilde{\mathcal{H}})$ . Let  $\tilde{\mathcal{H}}$  be a Hilbert space and denote by  $\tilde{\mathcal{H}}^n$  the  $n$ -fold tensor product  $\tilde{\mathcal{H}}^n = \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}} \otimes \dots \otimes \tilde{\mathcal{H}}$ . We set  $\tilde{\mathcal{H}}^0 = \mathbb{C}$  and  $F(\tilde{\mathcal{H}}) = \bigoplus_{n=0}^{\infty} \tilde{\mathcal{H}}^n$ . The  $F(\tilde{\mathcal{H}})$  is called the Fo'ck space over  $\tilde{\mathcal{H}}$ ; it will be separably, if  $\tilde{\mathcal{H}}$  is. For example, if  $\tilde{\mathcal{H}} = L_2(R)$ , then an element  $F(\tilde{\mathcal{H}})$  is a sequence of functions  $\psi = (\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), \dots)$ , so that  $|\psi_0|^2 + \sum_{n=1}^{\infty} |\psi_n(x_1, x_2, \dots, x_n)|^2 dx_1 dx_2 \dots dx_n < \infty$ .

Actually, it is not  $F(\tilde{\mathcal{H}})$ , itself, but two of its subspaces which are used most frequently in quantum field theory. These two subspaces are constructed as follows: Let  $P_n$  be the permutation group on  $n$  elements, and let  $\psi_n$  be a basis for space  $\tilde{\mathcal{H}}$ . For each  $\sigma \in P_n$ , we define an operator (which we also denote by  $\sigma$ ) on basis elements  $\tilde{\mathcal{H}}^n$  by  $\sigma(\varphi_{k_1} \otimes \varphi_{k_2} \otimes \dots \otimes \varphi_{k_n}) = \varphi_{k_{\sigma(1)}} \otimes \varphi_{k_{\sigma(2)}} \otimes \dots \otimes \varphi_{k_{\sigma(n)}}$ . The operator  $\sigma$  extends by linearity to a bounded operator (of norm one) on space  $\tilde{E}^n$ , so we can define  $S_n = \frac{1}{n!} \sum_{\sigma \in P_n} \sigma$ . That the operator  $S_n$  is the operator of orthogonal projection:  $S_n^2 = S_n$ , and  $S_n^* = S_n$ . The range of  $S_n$  is called  $n$ -fold symmetric tensor product of  $\tilde{\mathcal{H}}$ . In the case, where  $\tilde{\mathcal{H}} = L_2(R)$  and  $\tilde{\mathcal{H}}^n = L_2(R) \otimes L_2(R) \otimes \dots \otimes L_2(R) = L_2(R^n)$ ,  $S_n(\tilde{\mathcal{H}}^n)$  is just the subspace of  $L_2(R^n)$ , of all functions, left invariant under any permutation of the variables. We now define  $F_s(\tilde{\mathcal{H}}) = \bigoplus_{n=0}^{\infty} S_n(\tilde{\mathcal{H}}^n)$ . The space  $F_s(\tilde{\mathcal{H}})$  is called the symmetrical Fo'ck space over  $\tilde{\mathcal{H}}$ , or Boson Fo'ck space over  $\tilde{\mathcal{H}}$ .

Let  $\varepsilon(\cdot)$  is function from  $P_n$  to  $\{1, -1\}$ , which is one on even permutations and minus one on odd permutations. Define  $A_n = \frac{1}{n!} \sum_{\sigma \in P_n} \varepsilon(\sigma) \sigma$ ; then  $A_n$  is an orthogonal projector on  $\tilde{\mathcal{H}}^n$ .  $A_n(\tilde{\mathcal{H}}^n)$  is called the  $n$ -fold antisymmetrical tensor product of  $\tilde{\mathcal{H}}$ . In the case, where  $\tilde{\mathcal{H}} = L_2(R)$ ,  $A_n(\tilde{\mathcal{H}}^n)$  is just the subspace of  $L_2(R^n)$ , consisting of those functions odd under interchange of two coordinates. The subspace  $F_a(\tilde{\mathcal{H}}) = \bigoplus_{n=0}^{\infty} A_n(\tilde{\mathcal{H}}^n)$  is called the antisymmetrical Fo'ck space over  $\tilde{\mathcal{H}}$  or the Fermion Fo'ck space over  $\tilde{\mathcal{H}}$ .

## 2. Sextet state

Let  $\varphi_0$  be the vacuum vector in the space  $\tilde{\mathcal{H}}_{as}$ . The sextet state corresponds to the free motion of five electrons over the lattice with the basic functions  $s_{p,q,r,t,l \in Z^\nu}^{5/2} = a_{p,\uparrow}^+ a_{q,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ . The subspace  $\mathcal{H}_{5/2}^s$ , corresponding to the sextet state is the set of all vectors of the form  $\psi_{5/2}^s = \sum_{p,q,r,t,l \in Z^\nu} f(p, q, r, t, l) s_{p,q,r,t,l \in Z^\nu}^{\frac{5}{2}}, f \in l_2^{as}$ , where  $l_2^{as}$  is the subspace of antisymmetric functions in the space  $l_2((Z^\nu)^5)$ .

**Theorem 1.** *The subspace  $\mathcal{H}_{5/2}^s$  is invariant under the operator  $H$ , and the restriction  $H_{5/2}^s$  of operator  $H$  to the subspace  $\mathcal{H}_{5/2}^s$  is a bounded self-adjoint operator. It generates a bounded self-adjoint operator  $\bar{H}_{5/2}^s$  acting in the space  $l_2^{as}$  as  $\bar{H}_{5/2}^s \psi_{5/2}^s = 5A f(p, q, r, t, l) + B \sum_\tau [f(p + \tau, q, r, t, l) + f(p, q + \tau, r, t, l) + f(p, q, r + \tau, t, l) + f(p, q, r, t + \tau, l) + f(p, q, r, t, l + \tau)$*

*The operator  $H_{5/2}^s$  acts on a vector  $\psi_{5/2}^s \in \mathcal{H}_{5/2}^s$  as  $H_{5/2}^s \psi_{5/2}^s = \sum_{p,q,r,t,l \in Z^\nu} (\bar{H}_{5/2}^s f)(p, q, r, t, l) s_{p,q,r,t,l \in Z^\nu}^{\frac{5}{2}}$*

(4)

**Proof.** We act with the Hamiltonian  $H$  on vectors  $\psi_{5/2}^s \in \mathcal{H}_{5/2}^s$  using the standard anticommutation relations between electron creation and annihilation operators at lattice sites,  $\{a_{m,\gamma}, a_{n,\beta}^+\} = \delta_{m,n} \delta_{\gamma,\beta}$ ,  $\{a_{m,\gamma}, a_{n,\beta}\} = \{a_{m,\gamma}^+, a_{n,\beta}^+\} = \theta$ , and also take into account that  $a_{m,\gamma} \varphi_0 = \theta$ , where  $\theta$  is the zero element of  $\mathcal{H}_{5/2}^s$ . This yields the statement of the theorem.

**Lemma 1.** *The spectra of the operators  $H_{5/2}^s$  and  $\bar{H}_{5/2}^s$  coincide.*

**Proof.** Because the operators  $H_{5/2}^s$  and  $\bar{H}_{5/2}^s$  are bounded self-adjoint operators, it follows that if  $\lambda \in \sigma\left(H_{5/2}^s\right)$ , then the Weyl criterion (see [14], chapter VII, paragraph 3, pp. 262- 263) implies that there is a sequence  $\{\psi_n\}_{n=1}^\infty$  such that  $\|\psi_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|(H_{5/2}^s - \lambda)\psi_n\| = 0$ . We set  $\psi_n = \sum_{p,q,r,t,l} f_n(p, q, r, t, l) a_{p,\uparrow}^+ a_{q,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ . Then  $\|(H_{5/2}^s - \lambda)\psi_n\|^2 = \left( (H_{5/2}^s - \lambda)\psi_n, (H_{5/2}^s - \lambda)\psi_n \right) = \sum_{p,q,r,t,l} \left| (H_{5/2}^s - \lambda) f_n(p, q, r, t, l) \right|^2 \times$   
 $\times (a_{p,\uparrow}^+ a_{q,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0, a_{p,\uparrow}^+ a_{q,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0) =$   
 $= \sum_{p,q,r,t,l} \left| (H_{5/2}^s - \lambda) F_n(p, q, r, t, l) \right|^2 \times (a_{l,\uparrow} a_{t,\uparrow} a_{r,\uparrow} a_{q,\uparrow} a_{p,\uparrow} a_{p,\uparrow}^+ a_{q,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0, \varphi_0) =$   
 $= \sum_{p,q,r,t,l} \left| (H_{5/2}^s - \lambda) F_n(p, q, r, t, l) \right|^2 (\varphi_0, \varphi_0) =$   
 $= \sum_{p,q,r,t,l} \left| (H_{5/2}^s - \lambda) F_n(p, q, r, t, l) \right|^2 \rightarrow 0$  as  $n \rightarrow \infty$ , where  $F_n = \sum_{p,q,r,t,l} f_n(p, q, r, t, l)$ . It follows that  $\lambda \in \sigma(\bar{H}_{5/2}^s)$ . Consequently,  $\sigma\left(H_{5/2}^s\right) \subset \sigma\left(\bar{H}_{5/2}^s\right)$ . Conversely, let  $\bar{\lambda} \in \sigma\left(\bar{H}_{5/2}^s\right)$ . Then, by the Weyl criterion, there is a sequence  $\{F_n\}_{n=1}^\infty$  such that  $\|F_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|(\bar{H}_{5/2}^s - \bar{\lambda})F_n\| = 0$ . Setting  $F_n = \sum_{p,q,r,t,l} f_n(p, q, r, t, l)$ ,  $\|F_n\| = (\sum_{p,q,r,t,l} |f_n(p, q, r, t, l)|^2)^{1/2}$ , we conclude that  $\|\psi_n\| = \|F_n\| = 1$  and  $\|(\bar{H}_{5/2}^s - \bar{\lambda})F_n\| = \|(\bar{H}_{5/2}^s - \bar{\lambda})\psi_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . This means that  $\bar{\lambda} \in \sigma(H_{5/2}^s)$  and hence  $\sigma(H_{5/2}^s) \subset \sigma(\bar{H}_{5/2}^s)$ . These two relations imply  $\sigma\left(H_{5/2}^s\right) = \sigma\left(\bar{H}_{5/2}^s\right)$ .

We call the operator  $H_{5/2}^s$  the five-electron sextet state operator.

Let  $F: L_2((Z^v)^5) \rightarrow L_2((T^v)^5) \equiv \tilde{\mathcal{H}}_{5/2}^s$  be the Fourier transform, where  $T^v$  is the  $v$ - dimensional torus endowed with the normalized Lebesgue measure  $d\lambda$ , i.e.  $\lambda(T^v) = 1$ . We set  $\tilde{H}_{5/2}^s = F\bar{H}_{5/2}^s F^{-1}$ . In the quasimomentum representation, the operator  $\bar{H}_{5/2}^s$  acts in the Hilbert space  $L_2^{as}((T^v)^5)$  where  $L_2^{as}$  is the subspace of antisymmetric functions in  $L_2((T^v)^5)$ .

**Theorem 2.** *The Fourier transform of operator  $\bar{H}_{5/2}^s$  is an operator  $\tilde{H}_{5/2}^s = F\bar{H}_{5/2}^s F^{-1}$  acting in the space  $\tilde{\mathcal{H}}_{5/2}^s$  be the formula  $\tilde{H}_{5/2}^s \psi_{5/2}^s = h(\lambda, \mu, \gamma, \theta, \eta) f(\lambda, \mu, \gamma, \theta, \eta)$ ,* (5)

where  $h(\lambda, \mu, \gamma, \theta, \eta) = 5A + 2B \sum_{i=1}^v [\cos \lambda_i + \cos \mu_i + \cos \gamma_i + \cos \theta_i + \cos \eta_i]$ .

The proof Theorem 2, is straightforward of (3) using the Fourier transformation.

It is obvious that the spectrum of operator  $\tilde{H}_{5/2}^s$  is purely continuous and coincides with the value set of the function  $h(\lambda, \mu, \gamma, \theta, \eta)$ , i.e.,  $\sigma\left(\tilde{H}_{5/2}^s\right) = \sigma_{cont}\left(\tilde{H}_{5/2}^s\right) = [5A - 10Bv, 5A + 10Bv]$ . Therefore, the sextet state spectrum is independent of the Coulomb interaction parameter  $U$  and is the set of energies of five noninteracting electrons moving in the crystal. This result is totally natural because the sextet state cannot contain states with two electrons at a site. Hence, in the sextet state, the spectrum of five-electron systems can be evaluated exactly and is purely continuous. The spectral problem that we consider here is a particular case of the problem of finding the spectrum of a system of  $N$  noninteracting electrons in a crystal lattice.

By Hund's rule, the minimum-energy state in an  $N$  – electron system is the state where all spins are directed upward, i.e., the state  $\uparrow\uparrow \cdots \uparrow$ . By the Pauli exclusion principle, this state cannot contain states with two electrons at one site. In this case, the spectrum of the system is independent of the Coulomb interaction parameter  $U$  and is the band energy of  $N$  noninteracting electrons moving in the crystal. The spectrum of the system is then purely continuous.

### 3. First quartet state

In the system exists four type quartet states. The quartet state corresponds to the basic functions  ${}^1q_{m,n,r,t,l \in Z^v}^{3/2} = a_{m,\downarrow}^+ a_{n,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ ,  ${}^2q_{m,n,r,t,l \in Z^v}^{3/2} = a_{m,\uparrow}^+ a_{n,\downarrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ ,  ${}^3q_{m,n,r,t,l \in Z^v}^{3/2} = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{r,\downarrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ ,  ${}^4q_{m,n,r,t,l \in Z^v}^{3/2} = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\downarrow}^+ a_{l,\uparrow}^+ \varphi_0$ .

The subspace  ${}^1\tilde{\mathcal{H}}_{3/2}^q$ , corresponding to the first five-electron quartet state is the set of all vectors of the form  ${}^1\psi_{3/2}^q = \sum_{m,n,r,t,l \in Z^v} f(m,n,r,t,l) {}^1q_{m,n,r,t,l \in Z^v}^{3/2}$ ,  $f \in l_2^{as}$ , where  $l_2^{as}$  is the subspace of antisymmetric functions in the space  $l_2((Z^v)^5)$ . The restriction  ${}^1H_{3/2}^q$  of  $H$  to the subspace  ${}^1\tilde{\mathcal{H}}_{3/2}^q$ , is called the five-electron first quartet state operator.

**Theorem 3.** *The subspace  ${}^1\tilde{\mathcal{H}}_{3/2}^q$  is invariant under the operator  $H$ , and the operator  ${}^1H_{3/2}^q$  is a bounded self-adjoint operator. It generates a bounded self-adjoint operator  ${}^1\bar{H}_{3/2}^q$  acting in the space  $l_2^{as}$  as*

$${}^1\bar{H}_{3/2}^q {}^1\psi_{3/2}^q = 5Af(m,n,r,t,l) + B \sum_{\tau} [f(m+\tau,n,r,t,l) + f(m,n+\tau,r,t,l) + f(m,n,r+\tau,t,l) + f(m,n,r,t+\tau,l) + f(m,n,r,t,l+\tau) + U[\delta m,n+\delta m,r+\delta m,t+\delta m,l]]f(m,n,r,t,l) \quad (6)$$

$$\begin{aligned} &\text{The operator } {}^1H_{3/2}^q \text{ acts on a vector } {}^1\psi_{3/2}^q \in {}^1\tilde{\mathcal{H}}_{3/2}^q \text{ as} \\ &{}^1H_{3/2}^q {}^1\psi_{3/2}^q = \sum_{m,n,r,t,l \in Z^v} ({}^1\bar{H}_{3/2}^q f)(m,n,r,t,l) {}^1\psi_{3/2}^q. \end{aligned} \quad (7)$$

**Proof.** The proof of Theorem 3 is analogous to proof of Theorem 1.

We set  ${}^1\bar{H}_{3/2}^q = F {}^1H_{3/2}^q F^{-1}$ . In the quasimomentum representation, the operator  ${}^1\bar{H}_{3/2}^q$  acts in the Hilbert space  $L_2^{as}((T^v)^5)$  as

$$\begin{aligned} &{}^1\bar{H}_{3/2}^q {}^1\psi_{3/2}^q = \{5A + 2B \sum_{i=1}^v [\cos \lambda_i + \cos \mu_i + \cos \gamma_i + \cos \theta_i + \cos \eta_i]\} \times \\ &\quad \times f(\lambda, \mu, \gamma, \theta, \eta) + U \int_{T^v} [f(s, \lambda + \mu - s, \gamma, \theta, \eta) + f(s, \mu, \lambda + \gamma - s, \theta, \eta) + \\ &\quad + f(s, \mu, \gamma, \lambda + \theta - s, \eta) + f(s, \mu, \gamma, \theta, \lambda + \eta - s)] ds, \end{aligned} \quad (8)$$

where  $L_2^{as}((T^v)^5)$  is the subspace of antisymmetric functions in  $L_2((T^v)^5)$ . Taking into account that the function  $f(\lambda, \mu, \gamma, \theta, \eta)$  is antisymmetric, and using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [15], we can verify that the operator  ${}^1\bar{H}_{3/2}^q$  can be represented in the form

$${}^1\bar{H}_{3/2}^q {}^1\psi_{3/2}^q = \tilde{H}_2^1(\lambda, \mu) \otimes I \otimes I + I \otimes \tilde{H}_2^2(\gamma, \theta) \otimes I + I \otimes I \otimes \tilde{H}_2^3(\lambda, \eta), \quad (9)$$

where  $(\tilde{H}_2^1 f)(\lambda, \mu) = \{2A + 2B \sum_{n=1}^v (\cos \lambda_i + \cos \mu_i)\} f(\lambda, \mu) + U \int_{T^v} f(s, \lambda + \mu - s) ds$ ,  $(\tilde{H}_2^2 f)(\gamma, \theta) = \{2A + 2B \sum_{n=1}^v (\cos \gamma_i + \cos \theta_i)\} f(\gamma, \theta) + U \int_{T^v} f(s, \lambda + \theta - s) ds$ ,  $(\tilde{H}_2^3 f)(\lambda, \eta) = \{A + 2B \sum_{n=1}^v \cos \eta_i\} f(\lambda, \eta) - U \int_{T^v} f(s, \lambda + \gamma - s) ds - U \int_{T^v} f(s, \lambda + \eta - s) ds$ , and  $I$  is the unit operator.

The spectrum of the operator  $A \otimes I + I \otimes B$ , where  $A$  and  $B$  are densely defined bounded linear operators, was studied in [16-18]. Explicit formulas were given there that express the essential spectrum  $\sigma_{ess}(A \otimes I + I \otimes B)$  and discrete spectrum  $\sigma_{disc}(A \otimes I + I \otimes B)$  of operator  $A \otimes I + I \otimes B$  in terms of the spectrum  $\sigma(A)$  and the discrete spectrum  $\sigma_{disc}(A)$  of  $A$  and in terms of the spectrum  $\sigma(B)$  and the discrete spectrum  $\sigma_{disc}(B)$  of  $B$ :

$$\sigma_{disc}(A \otimes I + I \otimes B) = \{\sigma(A) \setminus \sigma_{ess}(A) + \sigma(B) \setminus \sigma_{ess}(B)\} \cup \{\sigma_{ess}(A) + \sigma(B)\} \cup \{\sigma(A) + \sigma_{ess}(B)\}, \quad (10)$$

$$\sigma_{ess}(A \otimes I + I \otimes B) = (\sigma_{ess}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ess}(B)). \quad (11)$$

It is clear that  $\sigma(A \otimes I + I \otimes B) = \{\lambda + \mu: \lambda \in \sigma(A), \mu \in \sigma(B)\}$ .

Consequently, we must investigate the spectrum of the operators  $\tilde{H}_2^1$ ,  $\tilde{H}_2^2$ , and  $\tilde{H}_2^3$ . Let the total quasimomentum of the two-electron system  $\lambda + \mu = \Lambda_1$  be fixed. We let  $L_2(\Gamma_{\Lambda_1})$  denote the space of functions that are square integrable on the manifold  $\Gamma_{\Lambda_1} = \{(\lambda, \mu): \lambda + \mu = \Lambda_1\}$ . It is known [19] that the operator  $\tilde{H}_2^1$  and the space  $\tilde{\mathcal{H}}_2^1 \equiv L_2((T^v)^2)$  can be decomposed into a direct integral  $\tilde{H}_2^1 = \bigoplus_{T^v} \tilde{H}_{2\Lambda_1}^1 d\Lambda_1$ ,  $\tilde{\mathcal{H}}_2^1 = \bigoplus_{T^v} \tilde{\mathcal{H}}_{2\Lambda_1}^1 d\Lambda_1$  of operators  $\tilde{H}_{2\Lambda_1}^1$  and spaces  $\tilde{\mathcal{H}}_{2\Lambda_1}^1 \equiv L_2(\Gamma_{\Lambda_1})$  such that the spaces  $\tilde{\mathcal{H}}_{2\Lambda_1}^1$  are invariant under the operators  $\tilde{H}_2^1$  and each operator  $\tilde{H}_2^1$  acts in  $\tilde{\mathcal{H}}_{2\Lambda_1}^1$  according to the formula  $(\tilde{H}_{2\Lambda_1}^1 f_{\Lambda_1})(\lambda) = \{2A + 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - \lambda_i)\} f_{\Lambda_1}(\lambda) + U \int_{T^v} f_{\Lambda_1}(s) ds$ , where  $f_{\Lambda_1}(x) = f(x, \Lambda_1 - x)$ .

It is known that the continuous spectrum of  $\tilde{H}_{2\Lambda_1}^1$  is independent of the parameter  $U$  and consists of the intervals  $\sigma_{cont}(\tilde{H}_{2\Lambda_1}^1) = G_{\Lambda_1}^v = [m_{\Lambda_1}^v, M_{\Lambda_1}^v] = [2A - 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2}, 2A + 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2}]$ .

**Definition 1.** The eigenfunction  $\varphi_{\Lambda_1} \in L_2(T^v \times T^v)$  of the operator  $\tilde{H}_{2\Lambda_1}^1$  corresponding to an eigenvalue  $z_{\Lambda_1} \in G_{\Lambda_1}^v$  is called a bound state (BS) (antibound state (ABS)) of  $\tilde{H}_2^1$  with the quasi momentum  $\Lambda_1$  and the quantity  $z_{\Lambda_1}$  is called the energy of this state.

We consider the operator  $K_{\Lambda_1}$  acting the space  $\tilde{\mathcal{H}}_{2\Lambda_1}^1$  according to the formula

$$(K_{\Lambda_1}(z) f_{\Lambda_1})(x) = \int_{T^v} \frac{U}{2A + 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - t_i) - z} f_{\Lambda_1}(t) dt.$$

It is a completely continuous operator in  $\tilde{\mathcal{H}}_{2\Lambda_1}^1$  for  $z_{\Lambda_1} \in G_{\Lambda_1}^v = [2A - 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2}, 2A + 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2}]$ .

$$\text{We set } D_{\Lambda_1}^v(z) = 1 + U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{2A + 4B \sum_{n=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - t_i) - z}.$$

**Lemma 2.** A number  $z_0 \in G_{\Lambda_1}^v$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^1$  if and only if it is a zero of the function  $D_{\Lambda_1}^v(z)$  i.e.,  $D_{\Lambda_1}^v(z_0) = 0$ .

**Proof.** Let the number  $z_0 \in G_{\Lambda_1}^v$  be an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^1$ , and  $\varphi_{\Lambda_1}(x)$  be the corresponding eigenfunction, i.e.,  $\{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - \lambda_i)\} \varphi_{\Lambda_1}(\lambda) + U \int_{T^v} \varphi_{\Lambda_1}(s) ds = z_0 \varphi_{\Lambda_1}(\lambda)$ .

Let  $\psi_{\Lambda_1}(x) = [2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - \lambda_i) - z] \varphi_{\Lambda_1}(x)$ . Then  $\psi_{\Lambda_1}(x) + U \int_{T^v} \frac{1}{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - s_i) - z} \psi_{\Lambda_1}(s) ds = 0$ , i.e., the number  $\mu = 1$  is an eigenvalue of the operator  $K_{\Lambda_1}(z)$ . It then follows that  $D_{\Lambda_1}^v(z_0) = 0$ . Now let  $z = z_0$  be a zero of the function  $D_{\Lambda_1}^v(z)$ , i.e.,  $D_{\Lambda_1}^v(z_0) = 0$ . It follows from the Fredholm theorem than the homogeneous equation  $\psi_{\Lambda_1}(x) + U \int_{T^v} \frac{1}{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_1^i}{2} \cos(\frac{\Lambda_1^i}{2} - s_i) - z} \psi_{\Lambda_1}(s) ds = 0$  has a nontrivial solution. This means that the number  $z = z_0$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^1$ .

We consider the one-dimensional case.

**Theorem 4.** a). At  $v=1$  and  $U < 0$ , and for all values of parameters of the Hamiltonian, the operator  $\tilde{H}_{2\Lambda_1}^1$  has a unique eigenvalue  $z_1 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ , that is below the continuous spectrum of  $\tilde{H}_{2\Lambda_1}^1$ , i.e.,  $z_1 < m_{\Lambda_1}^1$ .

b). At  $v=1$  and  $U > 0$ , and for all values of parameters of the Hamiltonian, the operator  $\tilde{H}_{2\Lambda_1}^1$  has a unique eigenvalue  $\tilde{z}_1 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ , that is above the continuous spectrum of  $\tilde{H}_{2\Lambda_1}^1$ , i.e.,  $\tilde{z}_1 > M_{\Lambda_1}^1$ .

**Proof.** If  $U < 0$ , then in the one-dimensional case, the function  $D_{\Lambda_1}^v(z)$  decreases monotonically outside the continuous spectrum domain of the operator  $\tilde{H}_{2\Lambda_1}^1$ , i.e., in the intervals  $(-\infty, m_{\Lambda_1}^1)$  and  $(M_{\Lambda_1}^1, +\infty)$ . For  $z < m_{\Lambda_1}^1$ , the function  $D_{\Lambda_1}^v(z)$  decreases from 1 to  $-\infty$ ,  $D_{\Lambda_1}^v(z) \rightarrow 1$  as  $z \rightarrow -\infty$ ,  $D_{\Lambda_1}^v(z) \rightarrow -\infty$ , as  $z \rightarrow m_{\Lambda_1}^1 - 0$ . Therefore, below the value  $m_{\Lambda_1}^1$ , the function  $D_{\Lambda_1}^v(z)$  has a single zero at the point  $z = z_1 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}} < m_{\Lambda_1}^1$ . For  $z > M_{\Lambda_1}^1$ , and  $U < 0$ , the function  $D_{\Lambda_1}^v(z)$  decreases from  $+\infty$  to 1,  $D_{\Lambda_1}^v(z) \rightarrow +\infty$ , as  $z \rightarrow M_{\Lambda_1}^1 + 0$ ,  $D_{\Lambda_1}^v(z) \rightarrow 1$  as  $z \rightarrow +\infty$ . Therefore, above the value  $M_{\Lambda_1}^1$ , the function  $D_{\Lambda_1}^v(z)$  cannot vanish. If  $U > 0$ , and  $z < m_{\Lambda_1}^1$ , the function  $D_{\Lambda_1}^v(z)$  increases from 1 to  $+\infty$ ,  $D_{\Lambda_1}^v(z) \rightarrow 1$  as

$z \rightarrow -\infty$ ,  $D_{\Lambda_1}^1(z) \rightarrow +\infty$  as  $z \rightarrow m_{\Lambda_1}^1 - 0$ . Therefore, below the value  $m_{\Lambda_1}^1$ , the function  $D_{\Lambda_1}^1(z)$  cannot vanish. For  $z > M_{\Lambda_1}^1$  and  $U > 0$ , the function  $D_{\Lambda_1}^1(z)$  increases from  $-\infty$  to 1,  $D_{\Lambda_1}^1(z) \rightarrow 1$  as  $z \rightarrow +\infty$ ,  $D_{\Lambda_1}^1(z) \rightarrow -\infty$  as  $z \rightarrow M_{\Lambda_1}^1 + 0$ . Therefore, above the value  $M_{\Lambda_1}^1$ , the function  $D_{\Lambda_1}^1(z)$  vanishes on a single point  $z = \tilde{z}_1 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ . In the two-dimensional case, we have analogous results. If  $U < 0$ , then the function  $D_{\Lambda_1}^2(z)$  decreases monotonically outside the continuous spectrum domain of the operator  $\tilde{H}_{2\Lambda_1}^1$ . For  $z < m_{\Lambda_1}^2$  the function  $D_{\Lambda_1}^2(z)$  decreases from 1 to  $-\infty$ ,  $D_{\Lambda_1}^2(z) \rightarrow 1$  as  $z \rightarrow -\infty$ ,  $D_{\Lambda_1}^2(z) \rightarrow -\infty$  as  $z \rightarrow m_{\Lambda_1}^2 - 0$ . Therefore, below the value  $m_{\Lambda_1}^2$ , the function  $D_{\Lambda_1}^2(z)$  has a single zero at the point  $z_1 < m_{\Lambda_1}^2$ . If  $U < 0$ , and  $z > M_{\Lambda_1}^2$ , then the function  $D_{\Lambda_1}^2(z)$  decreases from  $+\infty$  to 1. Therefore, above the value  $M_{\Lambda_1}^2$  the function  $D_{\Lambda_1}^2(z)$  cannot vanish. For  $U > 0$ , and  $z < m_{\Lambda_1}^2$ , the function  $D_{\Lambda_1}^2(z)$  increases from 1 to  $+\infty$ ,  $D_{\Lambda_1}^2(z) \rightarrow 1$  as  $z \rightarrow -\infty$ ,  $D_{\Lambda_1}^2(z) \rightarrow +\infty$  as  $z \rightarrow m_{\Lambda_1}^2 - 0$ . Therefore, below the value  $m_{\Lambda_1}^2$  the function  $D_{\Lambda_1}^2(z)$  cannot vanish. For  $U > 0$ , and  $z > M_{\Lambda_1}^2$ , the function  $D_{\Lambda_1}^2(z)$  increases from  $-\infty$  to 1,  $D_{\Lambda_1}^2(z) \rightarrow 1$  as  $z \rightarrow +\infty$ ,  $D_{\Lambda_1}^2(z) \rightarrow -\infty$  as  $z \rightarrow M_{\Lambda_1}^2 + 0$ . Therefore, above the value  $M_{\Lambda_1}^2$  the function  $D_{\Lambda_1}^2(z)$  has a single zero at the point  $\tilde{z}_1 > M_{\Lambda_1}^2$ .

We consider three-dimensional case. Denote  $m = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos \frac{\Lambda_i}{2} (1 + \cos(\frac{\Lambda_i}{2} - s_i))}$ .

For  $U < 0$ , and  $U < -\frac{4B}{m}$  below the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^1$  the function  $D_{\Lambda_1}^3(z)$  has a single zero at the point  $z_1 < m_{\Lambda_1}^3$ . For  $U < 0$ , and  $-\frac{4B}{m} \leq U < 0$ , below of the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^1$ , the function  $D_{\Lambda_1}^3(z)$  cannot vanish. We now denote  $M = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\sum_{i=1}^3 \cos \frac{\Lambda_i}{2} (1 - \cos(\frac{\Lambda_i}{2} - s_i))}$ . For  $U > 0$ , and  $U > \frac{4B}{M}$ , above the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^1$  the function  $D_{\Lambda_1}^3(z)$  has a single zero at the point  $\tilde{z}_1 > M_{\Lambda_1}^3$ . For  $U > 0$ , and  $0 < U \leq \frac{4B}{M}$ , above the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^1$  the function  $D_{\Lambda_1}^3(z)$  cannot vanish.

Consequently, we have the following theorem:

**Theorem 5.** a). If  $\nu = 3$  and  $U < 0$ ,  $U < -\frac{4B}{m}$ , then the operator  $\tilde{H}_{2\Lambda_1}^1$  has a unique eigenvalue  $z_1$ , the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ , i.e.,  $z_1 < m_{\Lambda_1}^3$ .

b). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{4B}{m} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_1}^1$  has no eigenvalue the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ .

c). If  $\nu = 3$  and  $U > 0$ ,  $U > \frac{4B}{M}$ , then the operator  $\tilde{H}_{2\Lambda_1}^1$  has a unique eigenvalue  $\tilde{z}_1$ , the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ , i.e.,  $\tilde{z}_1 > M_{\Lambda_1}^3$ .

d). If  $\nu = 3$  and  $U > 0$ ,  $0 < U \leq \frac{4B}{M}$ , then the operator  $\tilde{H}_{2\Lambda_1}^1$  has no eigenvalue the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ .

We let  $\Lambda_2 = \gamma + \theta$ . We now investigated the spectrum of the operator  $\tilde{H}_{2\Lambda_2}^2$ , i.e., the operator

$$(\tilde{H}_{2\Lambda_2}^2 f_{\Lambda_2})(\gamma) = \{2A + 4B \sum_{n=1}^{\nu} \cos \frac{\Lambda_2^i}{2} \cos(\frac{\Lambda_2^i}{2} - \gamma_i)\} f_{\Lambda_2}(\gamma) + U \int_{T^{\nu}} f_{\Lambda_2}(s) ds.$$

It is known that the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_2}^2$  is independent of  $U$  and coincides with the segment  $\sigma_{cont}(\tilde{H}_{2\Lambda_2}^2) = G_{\Lambda_2}^{\nu} = [m_{\Lambda_2}^{\nu}, M_{\Lambda_2}^{\nu}] = [2A - 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_2^i}{2}, 2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_2^i}{2}]$ .

Comparing the actions of operators  $\tilde{H}_{2\Lambda_1}^1$  and  $\tilde{H}_{2\Lambda_2}^2$ , we show that the operators  $\tilde{H}_{2\Lambda_2}^2$  and  $\tilde{H}_{2\Lambda_2}^2$  are the identical operators. Therefore, the spectra of these operators coincide. It is necessary only exchange  $\Lambda_1$  on  $\Lambda_2$ . We let  $z_2$  and  $\tilde{z}_2$  denote the eigenvalues of operator  $\tilde{H}_{2\Lambda_2}^2$ .

Let  $\Lambda_3 = \lambda + \eta$ . We now investigated the spectra of operator  $\tilde{H}_{\Lambda_3}^3$ .

$$(\tilde{H}_{\Lambda_3}^3 f_{\Lambda_3})(\lambda) = \{A + 2B \sum_{i=1}^{\nu} \cos(\Lambda_3 - \lambda_i)\} f_{\Lambda_3}(\lambda) - 2U \int_{T^{\nu}} f_{\Lambda_3}(s) ds.$$

It is known that the continuous spectrum of the operator  $\tilde{H}_{\Lambda_3}^3$  is independent of  $U$  and coincides with the segment  $\sigma_{cont}(\tilde{H}_{\Lambda_3}^3) = G_{\Lambda_3}^{\nu} = [m_{\Lambda_3}^{\nu}, M_{\Lambda_3}^{\nu}] = [A - 2B\nu, A + 2B\nu]$ .



Denote  $D_{\Lambda_3}^v = 1 - 2U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{A+2B \sum_{i=1}^v \cos(\Lambda_3^i - s_i) - z}$ .

The analogue of the Lemma 2 holds for the in this case. We consider the one-dimensional case.

**Theorem 6.** a). At  $v = 1$  and  $U < 0$ , and for all values of parameters of the Hamiltonian, the operator  $\tilde{H}_{\Lambda_3}^3$  has a unique eigenvalue  $\tilde{z}_3 = A + \sqrt{U^2 + B^2}$ , that is above the continuous spectrum of  $\tilde{H}_{\Lambda_3}^3$  i.e.,  $\tilde{z}_3 > M_{\Lambda_3}^3$ .

b). At  $v = 1$  and  $U > 0$ , and for all values of parameters of the Hamiltonian, the operator  $\tilde{H}_{\Lambda_3}^3$  has a unique eigenvalue  $z_3 = A - \sqrt{U^2 + B^2}$ , that is below the continuous spectrum of  $\tilde{H}_{\Lambda_3}^3$ , i.e.,  $z_3 < m_{\Lambda_3}^3$ .

We consider three-dimensional case, and the Watson integral  $W = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{3dx dy dz}{3 - \cos x - \cos y - \cos z} \approx 1,516$  (see. [20]).

Because the measure  $v$  is normalized, therefore  $\int_{T^3} \frac{3dx dy dz}{3 - \cos x - \cos y - \cos z} = \frac{W}{3}$ .

**Theorem 7.** a). At  $v = 3$ ,  $U < 0$ , and  $U < -\frac{3B}{W}$ , then the operator  $\tilde{H}_{\Lambda_3}^3$  has a unique eigenvalue  $\tilde{z}_3$ , that is above the continuous spectrum of  $\tilde{H}_{\Lambda_3}^3$ , i.e.,  $\tilde{z}_3 > M_{\Lambda_3}^3$ . If  $U < 0$ , and  $-\frac{3B}{W} \leq U < 0$ , then the operator  $\tilde{H}_{\Lambda_3}^3$  has no eigenvalue, that is above the continuous spectrum of  $\tilde{H}_{\Lambda_3}^3$ .

b). At  $v = 3$ ,  $U > 0$ , and  $U > \frac{3B}{W}$ , then the operator  $\tilde{H}_{\Lambda_3}^3$  has a unique eigenvalue  $z_3$  that is below the continuous spectrum of  $\tilde{H}_{\Lambda_3}^3$ , i.e.,  $z_3 < m_{\Lambda_3}^3$ . If  $U > 0$ , and  $0 < U \leq \frac{3B}{W}$ , then the operator  $\tilde{H}_{\Lambda_3}^3$  has no eigenvalue, that is below the continuous spectrum of  $\tilde{H}_{\Lambda_3}^3$ .

We now using the obtaining results and the representation (9), we can describe the structure of essential spectrum and discrete spectrum of the operator of first five-electron quartet state:

**Theorem 8.** At  $v = 1$  and  $U < 0$ , the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_{3/2}^q$  is exactly the union of seven segments:  $\sigma_{ess} \left( {}^1\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + z_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + z_3, b + z_2 + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + z_1 + z_3, d + z_1 + z_3] \cup [e + z_1 + z_2, f + z_1 + z_2]$ . The discrete spectrum of operator  ${}^1\tilde{H}_{3/2}^q$  consists of no more than one point:  $\sigma_{disc} \left( {}^1\tilde{H}_{3/2}^q \right) = \{z_1 + z_2 + \tilde{z}_3\}$ , or  $\sigma_{disc} \left( {}^1\tilde{H}_{3/2}^q \right) = \emptyset$ .

Here and hereafter  $a = 2A - 4B \cos \frac{\Lambda_1}{2}$ ,  $b = 2A + 4B \cos \frac{\Lambda_1}{2}$ ,  $c = 2A - 4B \cos \frac{\Lambda_2}{2}$ ,  $d = 2A + 4B \cos \frac{\Lambda_2}{2}$ ,  $e = A - 2B$ ,  $f = A + 2B$ ,  $z_1 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ ,  $z_2 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_2}{2}}$ ,  $\tilde{z}_3 = A + 2\sqrt{U^2 + B^2}$ .

**Proof.** If  $v=1$  and  $U < 0$ , then the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^1$  consists of the interval  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_1}^1 \right) = [a, b] = [2A - 4B \cos \frac{\Lambda_1}{2}, 2A + 4B \cos \frac{\Lambda_1}{2}]$ , and the discrete spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$  consists of a single eigenvalue  $z_1 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ . The continuous spectrum of the operator  $\tilde{H}_{2\Lambda_2}^2$  consists of the interval  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_2}^2 \right) = [c, d] = [2A - 4B \cos \frac{\Lambda_2}{2}, 2A + 4B \cos \frac{\Lambda_2}{2}]$ , and the discrete spectrum of operator  $\tilde{H}_{2\Lambda_2}^2$  consists of a single eigenvalue  $z_2 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_2}{2}}$ . The continuous spectrum of the operator  $\tilde{H}_{\Lambda_3}^3$  consists of the interval  $\sigma_{cont} \left( \tilde{H}_{\Lambda_3}^3 \right) = [e, f] = [A - 2B, A + 2B]$ , and the discrete spectrum of operator  $\tilde{H}_{\Lambda_3}^3$  consists of a single eigenvalue  $\tilde{z}_3 = A + \sqrt{U^2 + B^2}$ . It follows from representation (9) that  $\sigma \left( {}^1\tilde{H}_{3/2}^q \right) = \{\lambda + \mu + \gamma : \lambda \in \sigma(\tilde{H}_{2\Lambda_1}^1), \mu \in \sigma(\tilde{H}_{2\Lambda_2}^2), \gamma \in \sigma(\tilde{H}_{\Lambda_3}^3)\}$ .

Therefore, the essential spectrum of the system first five-electron doublet state operator  ${}^1\tilde{H}_3^q$  consists of the union of seven segments:  $[a+c+e, b+d+f]$ , and  $[a+c+\tilde{z}_3, b+d+\tilde{z}_3]$ , and  $[a+e+z_2, b+f+z_2]$ , and  $[a+z_2+\tilde{z}_3, b+z_2+\tilde{z}_3]$ , and  $[c+e+z_1, d+f+z_1]$ , and  $[c+z_1+\tilde{z}_3, d+z_1+\tilde{z}_3]$ , and  $[e+z_1+\tilde{z}_3, f+z_1+\tilde{z}_3]$ , and the number  $z_1+z_2+\tilde{z}_3$  is the eigenvalue of this operator (the antibound state energy). If  $z_1+z_2+\tilde{z}_3 \in \sigma_{ess} \left( {}^1\tilde{H}_3^q \right)$ , then the number  $z_1+z_2+\tilde{z}_3$  lie in the discrete spectrum of operator  ${}^1\tilde{H}_3^q$ , if  $z_1+z_2+\tilde{z}_3 \in \sigma_{ess} \left( {}^1\tilde{H}_3^q \right)$ , then discrete spectrum of operator  ${}^1\tilde{H}_3^q$  is empty, i.e.  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

The following theorem is proved totally similarly to Theorem 8.

**Theorem 9.** At  $v=1$  and  $U > 0$  the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is exactly the union of seven segments:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + c + z_3, b + d + z_3] \cup [a + c + z_3, b + d + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + e + z_1, d + f + z_1] \cup [c + e + z_1, d + f + z_1]$ .

The discrete spectrum of operator  ${}^1\tilde{H}_3^q$  is consists of no more than one point:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \{\tilde{z}_1 + \tilde{z}_2 + z_3\}$ , or  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

Here  $\tilde{z}_1 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ ,  $\tilde{z}_2 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_2}{2}}$ ,  $z_3 = A - 2\sqrt{U^2 + B^2}$ .

In the two-dimensional case we have the analogous results. We now consider the three-dimensional case. Let  $v=3$ :

**Theorem 10.** The following statements hold:

a). Let  $v=3$  and  $U < 0$ ,  $U < -\frac{4B}{m}$ ,  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $m > \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is the union of seven segments:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + c + z_3, b + d + z_3] \cup [a + c + z_3, b + d + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + e + z_1, d + f + z_1] \cup [c + e + z_1, d + f + z_1]$ . The discrete spectrum of operator  ${}^1\tilde{H}_3^q$  is consists of no more than one point:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \{z_1 + z_2 + \tilde{z}_3\}$ , or  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

Here,  $a = 2A - 4B \sum_{i=1}^3 \cos \frac{\Lambda_i}{2}$ ,  $b = 2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_i}{2}$ ,  $c = 2A - 4B \sum_{i=1}^3 \cos \frac{\Lambda_i}{2}$ ,  $d = 2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_i}{2}$ ,  $e = A - 6B$ ,  $f = A + 6B$ , and  $z_1$  is an eigenvalue of operator  $\tilde{H}_{2\Lambda_1}^1$ , and  $z_2$  is an eigenvalue of operator  $\tilde{H}_{2\Lambda_1}^2$ , and  $z_3$  is an eigenvalue of operator  $\tilde{H}_{\Lambda_3}^3$ .

b). Let  $v=3$ ,  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{m}$ , and  $m > \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is the union of four segments:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [c + e + z_1, d + f + z_1] \cup [e + z_1 + z_2, f + z_1 + z_2]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

c). Let  $v=3$ ,  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{3B}{W}$ , and  $m < \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is the union of two segments:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

d). Let  $v=3$ ,  $U < 0$ ,  $-\frac{4B}{m} \leq U < 0$  and  $m > \frac{4}{3}W$ , or  $-\frac{3B}{m} \leq U < 0$ , and  $m < \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of single segment:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f]$ , and the discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

**Theorem 11.** The following statements hold:

a). Let  $v=3$ ,  $U > 0$ , and  $U > \frac{4B}{M}$ ,  $M < \frac{4}{3}W$ , or  $U > 0$ , and  $U > \frac{3B}{W}$ ,  $M > \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is the union of seven segments:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + c + z_3, b + d + z_3] \cup [a + c + z_3, b + d + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + e + z_1, d + f + z_1] \cup [c + e + z_1, d + f + z_1]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \{\tilde{z}_1 + \tilde{z}_2 + z_3\}$ , or

$\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

Here,  $\tilde{z}_1$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^1$ , and  $\tilde{z}_2$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^2$ , and  $z_3$  is

an eigenvalue of the operator  $\tilde{H}_{\Lambda_3}^3$ .

b). Let  $v=3$ ,  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{3B}{W}$ , and  $M > \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is the union of four segments:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + \tilde{z}_2, b + f + z_2] \cup [c + e + z_1, d + f + z_1] \cup [e + z_1 + z_2, f + z_1 + z_2]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

c). Let  $v=3$ ,  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{M}$ , and  $M < \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is the union of two segments:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:

$$\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset.$$

d). Let  $v=3$ ,  $U > 0$ ,  $0 < U \leq \frac{3B}{W}$ , and  $M < \frac{4}{3}W$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of single segment:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a + c + e, b + d + f]$ , and the discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^1\tilde{H}_3^q \right) = \emptyset$ .

Let  $v = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ .

It is known that the continuous spectrum of  $\tilde{H}_{2\Lambda_1}^1$  is independent of  $U$  and coincides with the segment  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_1}^1 \right) = G_{\Lambda_1}^3 = \left[ 2A - 12B\cos\frac{\Lambda_1^0}{2}, 2A + 12B\cos\frac{\Lambda_1^0}{2} \right]$ .

**Theorem 12.** a). At  $v = 3$  and  $U < 0$  and the total quasimomentum  $\Lambda_1$  of the system have the form

$\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ . Then the operator  $\tilde{H}_{2\Lambda_1}^1$  has a unique eigenvalue  $z_1^1$ , if  $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , that is below the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ . Otherwise, the operator  $\tilde{H}_{2\Lambda_1}^1$  has no eigenvalue, that is below the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ .

b). At  $v=3$  and  $U > 0$ , and the total quasimomentum  $\Lambda_1$  of the system have the form  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ .

Then the operator  $\tilde{H}_{2\Lambda_1}^1$  has a unique eigenvalue  $z_1^2$ , if  $U > \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , that is above the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ . Otherwise, the operator  $\tilde{H}_{2\Lambda_1}^1$  has no eigenvalue, that is above the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^1$ .

It is known that the continuous spectrum of  $\tilde{H}_{2\Lambda_1}^2$  is independent of  $U$  and coincides with the segment  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_1}^2 \right) = G_{\Lambda_2}^3 = \left[ 2A - 12B\cos\frac{\Lambda_2^0}{2}, 2A + 12B\cos\frac{\Lambda_2^0}{2} \right]$ .

In this case, to take place the analogously theorem to theorem 12. It is necessary only exchange in this theorem  $\Lambda_1$  on  $\Lambda_2$ .

Now using the obtained results and representation (9), we describe the structure of the essential spectrum and the discrete spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$ .

Let  $v = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ .

**Theorem 13.** The following statements hold:

a). Let  $U < 0$ , and  $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^1\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \tilde{z}_3, b_1 + d_1 + \tilde{z}_3] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [a_1 + z_2^1 + \tilde{z}_3, b_1 + z_2^1 + \tilde{z}_3] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [c_1 + z_1^1 + \tilde{z}_3, d_1 + z_1^1 + z_3] \cup [e_1 + z_1 + z_2, f_1 + z_1 + z_2]$ .

The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \{z_1^1 + z_2^1 + \tilde{z}_3\}$ , or  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \emptyset$ .

Here and hereafter  $a_1 = 2A - 12B\cos\frac{\Lambda_1^0}{2}$ ,  $b_1 = 2A + 12B\cos\frac{\Lambda_1^0}{2}$ ,  $c_1 = 2A - 12B\cos\frac{\Lambda_2^0}{2}$ ,  $d = 2A + 12B\cos\frac{\Lambda_2^0}{2}$ ,  $e_1 = A - 6B$ ,  $f_1 = A + 6B$ , and  $z_1^1$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^1$ , and  $z_2^1$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^2$ , and  $\tilde{z}_3$  is an eigenvalue of the operator  $\tilde{H}_{\Lambda_3}^3$ .

b). Let  $U < 0$ , and  $-\frac{3B}{W} \leq U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of the union of four segments:  $\sigma_{ess}\left({}^1\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_1^1, b_1 + f_1 + z_2^1] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [e_1 + z_1^1 + z_2^1, f_1 + z_1^1 + z_2^1]$ . The discrete spectrum of the operator  ${}^1H32q$  is empty:  $\sigma_{disc}{}^1H32q = \emptyset$ .

c). Let  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of the union of two segments:  $\sigma_{ess}\left({}^1\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \tilde{z}_3, b_1 + d_1 + \tilde{z}_3]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \emptyset$ .

d). Let  $U < 0$ , and  $-\frac{3B}{W} \leq U < 0$ , and  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$  or  $-\frac{3B}{W} \leq U < 0$ , and  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of single segments:  $\sigma_{ess}\left({}^1\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$ , and the discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \emptyset$ .

**Theorem 14.** The following statements hold:

a). Let  $U > 0$ , and  $U > \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ , and  $U > \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of the union of seven segments:  $\sigma_{ess}\left({}^1\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [a_1 + z_2^2 + z_3, b_1 + z_2^2 + z_3] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2] \cup [c_1 + z_1^2 + z_3, d_1 + z_1^2 + z_3] \cup [e_1 + z_1^2 + z_2^2, f_1 + z_1^2 + z_2^2]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \{z_1^2 + z_2^2 + z_3\}$  or  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \emptyset$ .

Here and hereafter,  $z_1^2$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^1$ , and  $z_2^2$  is an eigenvalue of the operator  $\tilde{H}_{2\Lambda_1}^2$  and  $z_3$  is an eigenvalue of the operator  $\tilde{H}_{\Lambda_3}^3$ .

b). Let  $U > 0$ , and  $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < \frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ , and  $\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < \frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_3^q$  is consists of the union of four segments:  $\sigma_{ess}\left({}^1\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2] \cup [e_1 + z_1^2 + z_2^2, f_1 + z_1^2 + z_2^2]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^1\tilde{H}_3^q\right) = \emptyset$ .

c). Let  $U > 0$ , and  $\frac{3B}{W} \leq U < \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$  and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ , and  $\frac{3B}{W} \leq U < \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$  and  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_{\frac{3}{2}}^q$  is consists of the union of two segments:  $\sigma_{ess}\left({}^1\tilde{H}_{\frac{3}{2}}^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z3, b_1 + d_1 + z3]$ . The discrete spectrum of the operator  ${}^1\tilde{H}_{\frac{3}{2}}^q$  is empty:  $\sigma_{disc}\left({}^1\tilde{H}_{\frac{3}{2}}^q\right) = \emptyset$ .

d). Let  $U > 0$ ,  $0 < U < \frac{3B}{W}$ , and  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $0 < U < \frac{3B}{W}$  and  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ . Then the essential spectrum of the system first five-electron quartet state operator  ${}^1\tilde{H}_{\frac{3}{2}}^q$  is consists of single segments:  $\sigma_{ess}\left({}^1\tilde{H}_{\frac{3}{2}}^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$ , and the discrete spectrum of the operator  ${}^1\tilde{H}_{\frac{3}{2}}^q$  is empty:  $\sigma_{disc}\left({}^1\tilde{H}_{\frac{3}{2}}^q\right) = \emptyset$ .

#### 4. Second quartet state

The second quartet state corresponds the basic functions  ${}^2q_{m,n,r,t,l \in Z^v}^{3/2} = a_{m,\uparrow}^+ a_{n,\downarrow}^+ a_{r,\uparrow}^+ a_{t,\downarrow}^+ a_{l,\uparrow}^+ \varphi_0$ . The subspace  ${}^2\mathcal{H}_{3/2}^q$ , corresponding to the second five-electron quartet state is the set of all vectors of the form  ${}^2\psi_{3/2}^q = \sum_{m,n,r,t,l \in Z^v} f(m,n,r,t,l) {}^2q_{m,n,r,t,l \in Z^v}^{3/2}$ ,  $f \in l_2^{as}$  where  $l_2^{as}$  is the subspace of antisymmetric functions in the space  $l_2((Z^v)^5)$ .

The restriction  ${}^2H_{3/2}^q$  of operator  $H$  to the subspace  ${}^2\mathcal{H}_{3/2}^q$ , is called the five-electron second quartet state operator.

**Theorem 15.** The subspace  ${}^2\mathcal{H}_{3/2}^q$  is invariant under the operator  $H$ , and the operator  ${}^2H_{3/2}^q$  is a bounded self-adjoint operator. It generates a bounded self-adjoint operator  ${}^2\bar{H}_{3/2}^q$  acting in the space  $l_2^{as}$  as

$${}^2\bar{H}_{3/2}^q {}^2\psi_{3/2}^q = 5Af(m,n,r,t,l) + B \sum_{\tau} [f(m+\tau,n,r,t,l) + f(m,n+\tau,r,t,l) + f(m,n,r+\tau,t,l) + f(m,n,r,t+\tau,l) + f(m,n,r,t,l+\tau)] + U[\delta_{m,n} + \delta_{n,r} + \delta_{n,t} + \delta_{n,l}]f(m,n,r,t,l). \tag{12}$$

The operator  ${}^2H_{3/2}^q$  acts on a vector  ${}^2\psi_{3/2}^q \in {}^2\mathcal{H}_{3/2}^q$  as  ${}^2H_{3/2}^q {}^2\psi_{3/2}^q = \sum_{m,n,r,t,l \in Z^v} ({}^2\bar{H}_{3/2}^q f)(m,n,r,t,l) {}^2q_{3/2}^q$ . (13)

**Proof.** We act with the Hamiltonian  $H$  on vectors  ${}^2\psi_{3/2}^q \in {}^2\mathcal{H}_{3/2}^q$  using the standard anticommutation relations between electron creation and annihilation operators at lattice sites,  $\{a_{m,\gamma}, a_{n,\beta}^+\} = \delta_{m,n} \delta_{\gamma,\beta}$ ,  $\{a_{m,\gamma}, a_{n,\beta}\} = \{a_{m,\gamma}^+, a_{n,\beta}^+\} = \theta$ , and also take into account that  $a_{m,\gamma} \varphi_0 = \theta$ , where  $\theta$  is the zero element of  ${}^2\mathcal{H}_{3/2}^q$ . This yields the statement of the theorem.

We set  ${}^2\tilde{H}_{3/2}^q = F {}^2\bar{H}_{3/2}^q F^{-1}$ . In the quasimomentum representation, the operator  ${}^2\tilde{H}_{3/2}^q$  acts in the Hilbert space  $L_2^{as}((T^v)^5)$  as

$${}^2\tilde{H}_{3/2}^q {}^2\psi_{3/2}^q = \{5A + 2B \sum_{i=1}^v [\cos\lambda_i + \cos\mu_i + \cos\gamma_i + \cos\theta_i + \cos\eta_i] \times \\ \times f(\lambda, \mu, \gamma, \theta, \eta) + U \int_{T^v} [f(s, \lambda + \mu - s, \gamma, \theta, \eta) + f(\lambda, s, \mu + \gamma - s, \theta, \eta) + \\ + f(\lambda, s, \gamma, \mu + \theta - s, \eta) + f(\lambda, s, \gamma, \theta, \mu + \eta - s)] ds, \tag{14}$$

where  $L_2^{as}((T^v)^5)$  is the subspace of antisymmetric functions in  $L_2((T^v)^5)$ .

We verify that the operator  ${}^2\tilde{H}_{3/2}^q$  can be represented in the form

$${}^2\tilde{H}_{3/2}^q = \tilde{H}_2^4 \otimes I \otimes I + I \otimes \tilde{H}_2^5 \otimes I + I \otimes I \otimes \tilde{H}_2^6, \tag{15}$$

where  $(\tilde{H}_2^4 f)(\lambda, \mu) = \{2A + 2B \sum_{i=1}^v [\cos\lambda_i + \cos\mu_i]\} f(\lambda, \mu) + U \int_{T^v} f(s, \lambda + \mu - s) ds,$

$$(\widetilde{H}_2^5 f)(\gamma, \theta) = \left\{ 2A + 2B \sum_{i=1}^v [\cos \gamma_i + \cos \theta_i] \right\} f(\gamma, \theta) - U \int_{T^v} f(s, \mu + \theta - s) ds,$$

$$(\widetilde{H}_2^6 f)(\mu, \eta) = \{A + 2B \sum_{i=1}^v \cos \eta_i\} f(\lambda, \mu) + U \int_{T^v} f(s, \mu + \gamma - s) ds +$$

$$U \int_{T^v} f(s, \mu + \eta - s) ds.$$

Consequently, We must investigated the spectra of the operators  $\widetilde{H}_2^4$ ,  $\widetilde{H}_2^5$  and  $\widetilde{H}_2^6$ , the separately. The operator  $\widetilde{H}_2^4$  and  $\widetilde{H}_2^1$  is identical operators. Therefore, their spectrum is coincide. We use from this results.

Let  $\Lambda_2 = \gamma + \theta$ . We now investigated the spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$ :  $\{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_2^i}{2} \cos(\frac{\Lambda_2^i}{2} - \gamma_i - z)\} f \Lambda_2 \gamma - U T^v f \Lambda_2 s ds = 0$ .

It is clear that the continuous spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$  coincides with the segment

$$\sigma_{cont}(\widetilde{H}_{2\Lambda_2}^5) = G_{\Lambda_2}^v = [m_{\Lambda_2}^v, M_{\Lambda_2}^v] = [2A - 4B \sum_{i=1}^v \cos \frac{\Lambda_2^i}{2}, 2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_2^i}{2}].$$

$$\text{Let } D_{\Lambda_2}^v(z) = 1 - U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_2^i}{2} \cos(\frac{\Lambda_2^i}{2} - \gamma_i) - z}.$$

**Lemma 3.** The number  $z_0 \in G_{\Lambda_2}^v$  is an eigenvalue of operator  $\widetilde{H}_{2\Lambda_2}^5$  if and only if it is a zero of the function  $D_{\Lambda_2}^v(z)$ , i.e.,  $D_{\Lambda_2}^v(z_0) = 0$ .

It is clear that, if  $U < 0$  ( $U > 0$ ), then the exists only one solution of the equation  $D_{\Lambda_2}^v(z) = 0$ , the above (the below) of continuous spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$ .

First we consider the one-dimensional case. Let  $U < 0$ . Then the equation  $D_{\Lambda_2}^1(z) = 0$  in the above the continuous spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$  has a

only one solution  $z_2 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_2}{2}}$ . If  $U > 0$ , then the equation  $D_{\Lambda_2}^1(z) = 0$  in the below of the continuous spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$  has

a only one solution  $\tilde{z}_2 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_2}{2}}$ .

In the two-dimensional case we have the analogous situation. If  $U < 0$ , then the equation  $D_{\Lambda_2}^2(z) = 0$  in the above the continuous spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$  has a only one solution  $z_2 > M_{\Lambda_2}^2$ . If  $U > 0$ , then the equation  $D_{\Lambda_2}^2(z) = 0$  in the below the continuous spectrum of the operator  $\widetilde{H}_{2\Lambda_2}^5$  has a only one solution  $\tilde{z}_2 < m_{\Lambda_2}^2$ .

We now consider three-dimensional case. Denote  $M = \int_{T^3} \frac{ds_1 ds_2 ds_3}{2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_2^i}{2} (1 - \cos(\frac{\Lambda_2^i}{2} - s_i))}$ .

If  $U < -\frac{4B}{M}$ , then the above of the continuous spectrum of operator  $\widetilde{H}_{2\Lambda_2}^5$  the equation  $D_{\Lambda_2}^3(z) = 0$  have the only one solution  $z_2 > M_{\Lambda_2}^3$ . If  $-\frac{4B}{M} \leq U < 0$ , then the equation  $D_{\Lambda_2}^3(z) = 0$  has no solution in the above of continuous spectrum of operator  $\widetilde{H}_{2\Lambda_2}^5$ .

Denote  $m = \int_{T^3} \frac{ds_1 ds_2 ds_3}{2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_2^i}{2} (1 + \cos(\frac{\Lambda_2^i}{2} - s_i))}$ .

If  $U > 0$ ,  $U > \frac{4B}{m}$ , then the equation  $D_{\Lambda_2}^3(z) = 0$  has only one solution  $\tilde{z}_2 < m_{\Lambda_2}^3$ , lying in the below of continuous spectrum of operator  $\widetilde{H}_{2\Lambda_2}^5$ . If  $0 < U \leq \frac{4B}{m}$ , then the equation  $D_{\Lambda_2}^3(z) = 0$  has no solution in the below of continuous spectrum of operator  $\widetilde{H}_{2\Lambda_2}^5$ .

We now investigated the spectra of operator  $\widetilde{H}_2^6$ . Let  $\Lambda_3 = \mu + \eta$ .

$$(\widetilde{H}_{\Lambda_3}^6 f_{\Lambda_3})(\mu) = \{A + 2B \sum_{i=1}^v \cos(\Lambda_3 - \mu_i)\} f_{\Lambda_3}(\mu) + 2U \int_{T^v} f_{\Lambda_3}(s) ds.$$

It is known the continuous spectrum of operator  $\widetilde{H}_{\Lambda_3}^6$  is consists of interval  $\sigma_{cont}(\widetilde{H}_{\Lambda_3}^6) = G_{\Lambda_3}^v = [m_{\Lambda_3}^3, M_{\Lambda_3}^3] = [A - 2Bv, A + 2bv]$ .

Denote  $D_{\Lambda_3}^v(z) = 1 + 2U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{A + 2B \sum_{i=1}^v \cos(\Lambda_3 - s_i) - z}$ .

**Lemma 4.** The number  $z_0 \in G_{\Lambda_3}^\nu$  is an eigenvalue of operator  $\tilde{H}_{\Lambda_3}^6$  if and only if it is a zero of the function  $D_{\Lambda_3}^\nu(z)$ , i.e.,  $D_{\Lambda_3}^\nu(z_0) = 0$ .

It is known, at  $U > 0$  ( $U < 0$ ) there exists only one solution of the equation  $D_{\Lambda_3}^\nu(z) = 0$ , lying above (the below) of the continuous spectrum of the operator  $\tilde{H}_{\Lambda_3}^6$ .

We consider at first one-dimensional case. Let  $U < 0$ . The equation  $D_{\Lambda_3}^1(z) = 0$  has only one solution  $z_3 = A - 2\sqrt{U^2 + B^2}$ , lying below of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$ . If  $U > 0$ , then the equation  $D_{\Lambda_3}^1(z) = 0$  has only one solution  $\tilde{z}_3 = A + 2\sqrt{U^2 + B^2}$ , lying above of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$ .

In the two-dimensional case we have the analogous situation. If  $U < 0$ , the equation  $D_{\Lambda_3}^2(z) = 0$  has a unique solution  $z_2 < m_{\Lambda_3}^2$ , lying below of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$ . If  $U > 0$ , then the equation  $D_{\Lambda_3}^2(z) = 0$  has a unique solution  $\tilde{z}_2 > M_{\Lambda_3}^2$ , lying above of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$ .

We consider three-dimensional case.

If  $U < 0, U < -\frac{3B}{W}$ , then the equation  $D_{\Lambda_3}^3(z) = 0$  has only one solution  $\tilde{z}_3 < m_{\Lambda_3}^3$ , lying below of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$ . If  $-\frac{3B}{W} \leq U < 0$ , then the equation  $D_{\Lambda_3}^3(z) = 0$  in the below of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$  has no solution. If  $U > 0, U > \frac{3B}{W}$ , then the equation  $D_{\Lambda_3}^3(z) = 0$  has only one solution  $z_3 > M_{\Lambda_3}^3$ , lying above of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$ . If  $0 < U \leq \frac{3B}{W}$ , then the equation  $D_{\Lambda_3}^3(z) = 0$  in the above of the continuous spectrum of operator  $\tilde{H}_{\Lambda_3}^6$  has no solution.

We now using the obtaining results and representation (15), we can describe the structure of essential spectrum and discrete spectrum of the operator of second five-electron quartet state:

**Theorem 16.** If  $\nu = 1$  and  $U < 0$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_{3/2}^q$  consists of the union of seven segments:  $\sigma_{ess} \left( {}^2\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + \tilde{z}_3, b + z_2 + \tilde{z}_3] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [c + z_1 + z_3, d + z_1 + z_3] \cup [e + z_1 + z_2, f + z_1 + z_2]$ , and discrete spectrum of operator  ${}^2\tilde{H}_{3/2}^q$  consists of no more one point:  $\sigma_{disc} \left( {}^2\tilde{H}_{3/2}^q \right) = \{z_1 + \tilde{z}_2 + z_3\}$ , or  $\sigma_{disc} \left( {}^2\tilde{H}_{3/2}^q \right) = \emptyset$ .

Here and hereafter  $a = 2A - 4B \cos \frac{\Lambda_1}{2}$ ,  $b = 2A + 4B \cos \frac{\Lambda_1}{2}$ ,  $c = 2A - 4B \cos \frac{\Lambda_2}{2}$ ,  $d = 2A + 4B \cos \frac{\Lambda_2}{2}$ ,  $e = A - 2B$ ,  $f = A + 2B$ ,  $z_1 = 2A - \sqrt{U^2 + 16B^2 \cos \frac{\Lambda_1}{2}}$ ,  $\tilde{z}_2 = 2A + \sqrt{U^2 + 16B^2 \cos \frac{\Lambda_2}{2}}$ ,  $z_3 = A - 2\sqrt{U^2 + B^2}$ .

**Theorem 17.** If  $\nu = 1$  and  $U > 0$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_{3/2}^q$  consists of the union of seven segments:  $\sigma_{ess} \left( {}^2\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + \tilde{z}_3, b + z_2 + \tilde{z}_3] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [c + \tilde{z}_1 + \tilde{z}_3, d + \tilde{z}_1 + \tilde{z}_3] \cup [e + \tilde{z}_1 + z_2, f + \tilde{z}_1 + z_2]$ , and discrete spectrum of operator  ${}^2\tilde{H}_{3/2}^q$  consists of no more one point:  $\sigma_{disc} \left( {}^2\tilde{H}_{3/2}^q \right) = \{\tilde{z}_1 + z_2 + \tilde{z}_3\}$ , or  $\sigma_{disc} \left( {}^2\tilde{H}_{3/2}^q \right) = \emptyset$ .

Here  $\tilde{z}_1 = 2A + \sqrt{U^2 + 16B^2 \cos \frac{\Lambda_1}{2}}$ ,  $z_2 = 2A - \sqrt{U^2 + 16B^2 \cos \frac{\Lambda_2}{2}}$ ,  $\tilde{z}_3 = A + 2\sqrt{U^2 + B^2}$ .

In the two-dimensional case we have the analogous results.

We now consider the three-dimensional case.

**Theorem 18.** a). If  $\nu = 3$  and  $U < 0, U < \frac{4B}{m}, M > m, m < \frac{4}{3}W$ , or  $U < 0, U < -\frac{4B}{M}, m > M, m < \frac{4}{3}W$ , or  $U < 0, U < -\frac{3B}{W}, M > m, m > \frac{4}{3}W$ , or  $U < 0, U < -\frac{3B}{W}, m > M, M > \frac{4}{3}W$ , then the essential spectrum of the second

five-electron quartet state operator  ${}^2\tilde{H}_{3/2}^q$  consists of the union of seven segments:  $\sigma_{ess} \left( {}^2\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + \tilde{z}_2, b + f + \tilde{z}_2] \cup [a + \tilde{z}_2 + z_3, b + \tilde{z}_2 + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + z_1 + z_3, d + z_1 + z_3] \cup [e + z_1 + \tilde{z}_2, f + z_1 + \tilde{z}_2]$ , and discrete spectrum of operator  ${}^2\tilde{H}_{3/2}^q$  consists of

no more one point:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \{z_1 + \tilde{z}_2 + z_3\}$ , or  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

Here,  $a = 2A - 4B \sum_{i=1}^3 \cos \frac{\Lambda_i^1}{2}$ ,  $b = 2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_i^1}{2}$ ,  $c = 2A - 4B \sum_{i=1}^3 \cos \frac{\Lambda_i^2}{2}$ ,  $d = 2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_i^2}{2}$ ,  $e = A - 6B$ ,  $f = A + 6B$ ,  $z_1$ ,  $\tilde{z}_2$ , and  $z_3$ , are the eigenvalues of the operators  $\tilde{H}_2^4$ ,  $\tilde{H}_2^5$  and  $\tilde{H}_2^6$ , correspondingly.

b). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{m}$ ,  $M > m$ , and  $m > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{M}$ ,  $m > M$ , and  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{4B}{M}$ ,  $M > m$ , and  $M < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{4B}{m}$ ,  $M < m$ , and  $m < \frac{4}{3}W$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_{3/2}^q$  is consists of the union of four segments:

$$\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + \tilde{z}_2, b + f + \tilde{z}_2] \cup [c + e + z_1, d + f + z_1] \cup [z_1 + z_2, f + z_1 + z_2],$$

$$\text{or } \sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + \tilde{z}_2, b + f + \tilde{z}_2] \cup [a + \tilde{z}_2 + z_3, b + \tilde{z}_2 + z_3] \cup [c + e + z_3, d + f + z_3] \cup [z_1 + z_3, d + z_1 + z_3] \cup [z_1 + z_3, f + z_2 + z_3] \text{ or}$$

$$\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + z_1, b + f + z_1] \cup [c + e + z_3,$$

$$d + f + z_3] \cup [e + z_1 + z_3, f + z_1 + z_3], \text{ and discrete spectrum of operator } {}^2\tilde{H}_3^q \text{ is empty: } \sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset.$$

c). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{M}$ ,  $M > m$ ,  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{3B}{W}$ ,  $m > M$ , and  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{3B}{M}$ ,  $m < M$ , and  $M < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{4B}{M}$ ,  $m < M$ , and  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{4B}{m}$ ,  $m > M$ , and  $m > \frac{4}{3}W$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of two segments:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_2, b + d + z_2]$ , or  $\sigma_{ess} {}^2\tilde{H}_3^q = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3]$ , or

$$\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_1, b + d + z_1], \text{ and discrete spectrum of operator } {}^2\tilde{H}_3^q \text{ is empty:}$$

$$\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset.$$

d). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{4B}{M} \leq U < 0$ , and  $M > m$ ,  $M > \frac{4}{3}W$ , or  $-\frac{4B}{M} \leq U < 0$ , and  $M > m$ ,  $M > \frac{4}{3}W$ , or  $-\frac{3B}{m} \leq U < 0$ , and  $M > m$ ,  $M < \frac{4}{3}W$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is single segment:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:

$$\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset.$$

**Theorem 19.** a). If  $\nu = 3$  and  $U > 0$ ,  $U > \frac{3B}{W}$ ,  $m > \frac{4}{3}W$ , and  $m < M$ , or  $U > 0$ ,  $U > \frac{4B}{M}$ ,  $m < \frac{4}{3}W$ , and  $M < m$ , or  $U > 0$ ,  $U > \frac{4B}{m}$ ,  $m < \frac{4}{3}W$ , and  $m < M$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + z_3, b + z_2 + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + z_1 + z_3, d + z_1 + z_3] \cup [e + z_1 + z_2, f + z_1 + z_2]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc} {}^2\tilde{H}_3^q = \{z_1 + z_2 + z_3\}$ , or

$$\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset.$$

Here,  $\tilde{z}_1, z_2$  and  $\tilde{z}_3$ , are the eigenvalues of the operators  $\tilde{H}_{2\Lambda_1}^4$ ,  $\tilde{H}_{2\Lambda_2}^5$  and  $\tilde{H}_{\Lambda_3}^6$ , correspondingly.

b). If  $\nu = 3$  and  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{4B}{m}$ , and  $m < M$ ,  $m < \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{4B}{M}$ ,  $M < m$ , and



$m < \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{3B}{W}$ , and  $M < m$ ,  $m > \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{3B}{W}$ , and  $m < M$ ,  $m > \frac{4}{3}W$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of four segments:

$$\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [c + \tilde{z}_1 + \tilde{z}_3, d + \tilde{z}_1 + \tilde{z}_3],$$

$$\text{or } \sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + \tilde{z}_3, b + z_2 + \tilde{z}_3],$$

$$\text{or } \sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + z_2, b + f + z_2] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [e + \tilde{z}_1 + z_2, f + \tilde{z}_1 + z_2],$$

and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

c). If  $\nu = 3$  and  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{m}$ , and  $m < \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{m}$ , and  $M < \frac{4}{3}W$ ,  $m < M$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{3B}{W}$ ,  $M < m$ , and  $m > \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{3B}{W}$ , and  $M > \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{4B}{M}$ , and  $M > \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{4B}{m}$ , and  $m > \frac{4}{3}W$ ,  $m < M$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of two segments:

$$\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3],$$

$$\text{or } \sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + z_2, b + f + z_2],$$

$$\text{or } \sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1],$$

and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

d). If  $\nu = 3$  and  $U > 0$ ,  $0 < U \leq \frac{3B}{W}$ , and  $M < \frac{4}{3}W$ , or  $U > 0$ ,  $0 < U \leq \frac{3B}{W}$ , and  $m < \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $0 < U \leq \frac{4B}{m}$ ,  $M < m$ , and  $m > \frac{4}{3}W$ , or  $U > 0$ ,  $0 < U \leq \frac{4B}{M}$ , and  $m > \frac{4}{3}W$ ,  $M > m$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is single segment:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a + c + e, b + d + f]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

We now consider the three-dimensional case, when  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ . Then the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^4$  is consists of the segment  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_1}^4 \right) = G_{\Lambda_1}^3 = \left[ 2A - 12B\cos\frac{\Lambda_1^0}{2}, 2A + 12B\cos\frac{\Lambda_1^0}{2} \right]$ .

**Theorem 20.** a). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U < 0$ , and  $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_1}^4$  has only one eigenvalue  $z_1^1$ , lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^4$ .

b). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_1}^4$  has no eigenvalue, lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^4$ .

c). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U > 0$ , and  $U > \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_1}^4$  has only one eigenvalue  $z_1^2$ , lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^4$ .

d). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U > 0$ , and  $0 < U \leq \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_1}^4$  has no eigenvalue, lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^4$ .

We now consider the three-dimensional case, when  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ . Then the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_2}^5$  is consists of the segment  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_2}^5 \right) = G_{\Lambda_2}^3 = \left[ 2A - 12B\cos\frac{\Lambda_2^0}{2}, 2A + 12B\cos\frac{\Lambda_2^0}{2} \right]$ .

**Theorem 21.** a). If  $\nu = 3$  and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ , and  $U < 0$ , and  $U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_2}^5$  has only one eigenvalue  $z_2^1$ , lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^5$ .

b). If  $\nu = 3$  and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ , and  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_2}^5$  has no eigenvalue, lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^5$ .

c). If  $\nu = 3$  and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ , and  $U > 0$ , and  $U > \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_2}^5$  has only one eigenvalue  $z_2^2$ , lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^5$ .

d). If  $\nu = 3$  and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ , and  $U > 0$ , and  $0 < U \leq \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_2}^5$  has no eigenvalue, lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^5$ .

We now using the obtaining results and representation (15), we can describe the structure of essential spectrum and discrete spectrum of the operator of second five-electron quartet state:

Let  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_2 = (\Lambda_2^0, \Lambda_2^0, \Lambda_2^0)$ .

**Theorem 22.** a). If  $\nu = 3$  and  $U < 0$ ,  $U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is

consists of the union of seven segments:  $\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [a_1 + z_2^1 + z_3, b_1 + z_2^1 + z_3] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [c_1 + z_1^1 + z_3, d_1 + z_1^1 + z_3] \cup [e_1 + z_1^1 + z_3, f_1 + z_1^1 + z_3]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc}\left({}^2\tilde{H}_3^q\right) = \{z_1^1 + z_2^1 + z_3\}$ , or  $\sigma_{disc}\left({}^2\tilde{H}_3^q\right) = \emptyset$ .

Here, and hereafter  $a_1 = 2A - 12B\cos\frac{\Lambda_1^0}{2}$ ,  $b_1 = 2A + 12B\cos\frac{\Lambda_1^0}{2}$ ,  $c_1 = 2A - 12B\cos\frac{\Lambda_2^0}{2}$ ,  $d_1 = 2A + 12B\cos\frac{\Lambda_2^0}{2}$ ,  $e_1 = A - 6B$ ,  $f_1 = A + 6B$ , and  $z_1^1, z_2^1$ , and  $z_3$ , are the eigenvalues of the operators  $\tilde{H}_{2\Lambda_1}^4$ ,  $\tilde{H}_{2\Lambda_2}^5$  and  $\tilde{H}_{\Lambda_3}^6$ , correspondingly.

b). If  $\nu = 3$ , and  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{3B}{W} \leq U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{3B}{W} \leq U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union

of four segments:  $\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [e_1 + z_1^1 + z_3, f_1 + z_1^1 + z_3]$ , or

$\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \bar{z}_3, b_1 + d_1 + \bar{z}_3] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [c_1 + z_1^1 + \bar{z}_3, d_1 + z_1^1 + \bar{z}_3]$ , or  $\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [e_1 + z_1^1 + z_2^1, f_1 + z_1^1 + z_2^1]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^2\tilde{H}_3^q\right) = \emptyset$ .

c). If  $\nu = 3$ , and  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union

$\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U < -\frac{12B\cos \frac{A_1^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos \frac{A_1^0}{2}}{W} \leq U < -\frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of two segments:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1]$ , or  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1]$ , or  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + \tilde{z}_3, b_1 + d_1 + \tilde{z}_3]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

d). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{3B}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $-\frac{3B}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos \frac{A_1^0}{2}}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is single segment:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

**Theorem 23.** a). If  $\nu = 3$ , and  $U > 0$ ,  $U > \frac{12B\cos \frac{A_1^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $U > \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $U > \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $U > \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [a_1 + z_2^2 + z_3, b_1 + z_2^2 + z_3] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2] \cup [c_1 + z_1^2 + z_3, d_1 + z_1^2 + z_3] \cup [z_1^2 + z_2^2 + z_3]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \{z_1^2 + z_2^2 + z_3\}$ , or  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

Here,  $z_1^2$ ,  $z_2^2$  and  $z_3$ , are the eigenvalues of the operators  $\tilde{H}_{2A_1}^4$ ,  $\tilde{H}_{2A_2}^5$  and  $\tilde{H}_{A_3}^6$ , correspondingly.

b). If  $\nu = 3$ , and  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos \frac{A_1^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_1^0}{2}}{W} \leq U \leq \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ , and  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{12B\cos \frac{A_1^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{12B\cos \frac{A_1^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of four segments:  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2] \cup [c_1 + z_1^2 + z_3, d_1 + z_1^2 + z_3]$ , or  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [a_1 + z_2^2 + z_3, b_1 + z_2^2 + z_3]$ , or  $\sigma_{ess} \left( {}^2\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2] \cup [e_1 + z_1^2 + z_2^2, f_1 + z_1^2 + z_2^2]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^2\tilde{H}_3^q \right) = \emptyset$ .

c). If  $\nu = 3$ , and  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U \leq \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < \frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$  or  $U > 0$ ,  $\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U < \frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is consists of the union of two segments:  $\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2]$ , or  $\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + e_1 + z_1^2, b_1 + d_1 + f_1 + z_1^2]$ , or  $\sigma_{ess}2H32q = a1+c1+e1, b1+d1+f1 \cup a1+c1+z3, b1+d1+z3$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is

empty:  $\sigma_{disc}\left({}^2\tilde{H}_3^q\right) = \emptyset$ .

d). If  $\nu = 3$ , and  $U > 0$ ,  $0 < U < \frac{3B}{W}$ , and  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $0 < U < \frac{3B}{W}$ , and  $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_1^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $0 < U < \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , and  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $0 < U < \frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ , and  $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_1^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the second five-electron quartet state operator  ${}^2\tilde{H}_3^q$  is single segment:  $\sigma_{ess}\left({}^2\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$ , and discrete spectrum of operator  ${}^2\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^2\tilde{H}_3^q\right) = \emptyset$ .

### 5. Third quartet state

The third quartet state corresponds the basic functions  ${}^3q_{m,n,r,t,l \in Z^v}^{3/2} = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{r,\downarrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ . The subspace  ${}^3\mathcal{H}_{3/2}^q$ , corresponding to the third five-electron quartet state is the set of all vectors of the form  ${}^3\psi_{3/2}^q = \sum_{m,n,r,t,l \in Z^v} f(m,n,r,t,l) {}^3q_{m,n,r,t,l \in Z^v}^{3/2}$ ,  $f \in l_2^{as}$ , where  $l_2^{as}$  is the subspace of antisymmetric functions in the space  $l_2((Z^v)^5)$ .

The restriction  ${}^3H_{3/2}^q$  of operator  $H$  to the subspace  ${}^3\mathcal{H}_{3/2}^q$ , is called the five-electron third quartet state operator.

**Theorem 24.** The subspace  ${}^3\mathcal{H}_{3/2}^q$  is invariant under the operator  $H$ , and the operator  ${}^3H_{3/2}^q$  is a bounded self-adjoint operator. It generates a bounded self-adjoint operator  ${}^3\bar{H}_{3/2}^q$  acting in the space  $l_2^{as}$  as

$${}^3\bar{H}_{3/2}^q {}^3\psi_{3/2}^q = 5Af(m,n,r,t,l) + B \sum_{\tau} [f(m+\tau,n,r,t,l) + f(m,n+\tau,r,t,l) + f(m,n,r+\tau,t,l) + f(m,n,r,t+\tau,l) + f(m,n,r,t,l+\tau) + U[\delta m,r + \delta n,r + \delta r,t + \delta r,l]]. \quad (16)$$

$$\text{The operator } {}^3H_{3/2}^q \text{ acts on a vector } {}^3\psi_{3/2}^q \in {}^3\mathcal{H}_{3/2}^q \text{ as} \\ {}^3H_{3/2}^q {}^3\psi_{3/2}^q = \sum_{m,n,r,t,l \in Z^v} ({}^3\bar{H}_{3/2}^q f)(m,n,r,t,l) {}^3\psi_{3/2}^q. \quad (17)$$

**Proof.** We act with the Hamiltonian  $H$  on vectors  ${}^3\psi_{3/2}^q \in {}^3\mathcal{H}_{3/2}^q$  using the standard anticommutation relations between electron creation and annihilation operators at lattice sites,  $\{a_{m,\gamma}, a_{n,\beta}^+\} = \delta_{m,n} \delta_{\gamma,\beta}$ ,  $\{a_{m,\gamma}, a_{n,\beta}\} = \{a_{m,\gamma}^+, a_{n,\beta}^+\} = \theta$ , and also take into account that  $a_{m,\gamma} \varphi_0 = \theta$ , where  $\theta$  is the zero element of  ${}^3\mathcal{H}_{3/2}^q$ . This yields the statement of the theorem.

We set  ${}^3\tilde{H}_{3/2}^q = \mathcal{F} {}^3\bar{H}_{3/2}^q \mathcal{F}^{-1}$ . In the quasimomentum representation, the operator  ${}^3\tilde{H}_{3/2}^q$  acts in the Hilbert space  $L_2^{as}((T^v)^5)$  as

$${}^3\tilde{H}_{3/2}^q {}^3\psi_{3/2}^q = \left\{ 5A + 2B \sum_{i=1}^v [\cos\lambda_i + \cos\mu_i + \cos\gamma_i + \cos\theta_i + \cos\eta_i] \right\} \times$$

$$\begin{aligned} &\times f(\lambda, \mu, \gamma, \theta, \eta) + U \int_{T^v} [f(s, \mu, \lambda + \gamma - s, \theta, \eta) + f(\lambda, s, \mu + \gamma - s, \theta, \eta) + \\ &\quad + f(\lambda, \mu, s, \gamma + \theta - s, \eta) + f(\lambda, \mu, s, \theta, \gamma + \eta - s)] ds, \end{aligned} \tag{18}$$

where  $L_2^{as}((T^v)^5)$  is the subspace of antisymmetric functions in  $L_2((T^v)^5)$ .

We verify that the operator  ${}^3\tilde{H}_{3/2}^q$  can be represented in the form

$${}^3\tilde{H}_{3/2}^q = \tilde{H}_2^7 \otimes I \otimes I + I \otimes \tilde{H}_2^8 \otimes I + I \otimes I \otimes \tilde{H}_2^9, \tag{19}$$

Where

$$\begin{aligned} (\tilde{H}_2^7 f)(\lambda, \gamma) &= \{2A + 2B \sum_{i=1}^v [\cos \lambda_i + \cos \gamma_i] f(\lambda, \gamma) - U \int_{T^v} [f(s, \lambda + \gamma - s) ds, (\tilde{H}_2^8 f)(\mu, \gamma) = \{A + \\ 2Bi=1vcos\mu_i f(\mu, \gamma) + 2UTv[f(s, \mu + \gamma - s) ds, \quad H29f\theta, \eta &= 2A + 2Bi=1v[\cos\theta_i + \cos\eta_i f\theta, \eta - UTv[f(s, \gamma + \eta - s) ds. \end{aligned}$$

Consequently, We must investigated the spectra of the operators  $\tilde{H}_2^7$ ,  $\tilde{H}_2^8$  and  $\tilde{H}_2^9$ , the separately.

Now, we investigated the spectrum of operator  $\tilde{H}_2^7$ . Let  $\Lambda_1 = \lambda + \gamma$  be fixed. That the operator  $\tilde{H}_2^7$  and the space  $\tilde{\mathcal{H}}_2^7 \equiv L_2((T^v)^2)$  can be expanded into the direct integrals  $\tilde{H}_2^7 = \oplus \int_{T^v} \tilde{H}_{2\Lambda_1}^7 d\Lambda_1$ ,  $\tilde{\mathcal{H}}_2^7 = \oplus \int_{T^v} \tilde{\mathcal{H}}_{2\Lambda_1}^7 d\Lambda_1$ .

It is known that the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^7$  does not

depend on the parameter  $U$  and consists of the intervals  $\sigma_{cont}(\tilde{H}_{2\Lambda_1}^7) = G_{\Lambda_1}^v = \left[-2A - 4B \sum_{i=1}^v \cos \frac{\Lambda_1^i}{2}, -2A + 4Bi=1vcos\Lambda_1i2\right]$ .

$$\text{Let } D_{\Lambda_1}^v(z) = 1 - U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{-2A - 4B \sum_{i=1}^v \cos \frac{\Lambda_1^i}{2} \cos \left(\frac{\Lambda_1^i}{2} - s_i\right) - z}.$$

We have the next Lemma.

**Lemma 5.** The number  $z_0 \in G_{\Lambda_1}^v$  is an eigenvalue of operator  $\tilde{H}_{2\Lambda_1}^7$  if and only if it is a zero of the function  $D_{\Lambda_1}^v(z)$ , i.e.,  $D_{\Lambda_1}^v(z_0) = 0$ .

In the one-dimensional case we have the following theorems:

**Theorem 25.** Let  $v = 1$ . Then

a). If  $U > 0$ , then the operator  $\tilde{H}_{2\Lambda_1}^7$  has a unique eigenvalue  $z_1 = -2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ , lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

b). If  $U < 0$ , then the operator  $\tilde{H}_{2\Lambda_1}^7$  has a unique eigenvalue  $\tilde{z}_1 = -2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$ , lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

In the two-dimensional case, we have the analogous results.

We now consider three-dimensional case.

$$\text{We denote } M = \int_{T^3} \frac{ds_1 ds_2 ds_3}{2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_2^i}{2} (1 - \cos(\frac{\Lambda_2^i}{2} - s_i))} \text{ and } m = \int_{T^3} \frac{ds_1 ds_2 ds_3}{2A + 4B \sum_{i=1}^3 \cos \frac{\Lambda_2^i}{2} (1 + \cos(\frac{\Lambda_2^i}{2} - s_i))}.$$

**Theorem 26.** Let  $v = 3$ . Then

a). If  $U > \frac{4B}{M}$ , then operator  $\tilde{H}_{2\Lambda_1}^7$  has a unique eigenvalue  $z_1$ , lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

b). If  $0 < U \leq \frac{4B}{M}$ , then operator  $\tilde{H}_{2\Lambda_1}^7$  has no eigenvalues, lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

c). If  $U < -\frac{4B}{m}$ , then operator  $\tilde{H}_{2\Lambda_1}^7$  has a unique eigenvalue  $\tilde{z}_1$ , lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

d). If  $-\frac{4B}{m} \leq U < 0$ , then operator  $\tilde{H}_{2\Lambda_1}^7$  has no eigenvalues, lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

We now investigated the spectrum of operator

$$(\tilde{H}_2^8 f)(\mu, \gamma) = \{A + 2B \sum_{i=1}^v \cos \mu_i\} f(\mu, \gamma) + 2U \int_{T^v} [f(s, \mu + \gamma - s) ds.$$

Let  $\Lambda_2 = \mu + \gamma$  be fixed. Then  $(\tilde{H}_{2\Lambda_2}^8 f_{\Lambda_2})(\mu) = \{A + 2B \sum_{i=1}^v \cos \mu_i\} f_{\Lambda_2}(\mu) + 2U \int_{T^v} f_{\Lambda_2}(s) ds$ .

The continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$  is consists of interval  $G_{\Lambda_2}^v = [m_{\Lambda_2}^v, M_{\Lambda_2}^v] = [A - 2Bv, A + 2Bv]$ .

We set  $D_{\Lambda_2}^v(z) = 1 + 2U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{A + 2B \sum_{i=1}^v \cos s_i - z}$ . Then we have the next Lemma.

**Lemma 6.** The number  $z_0 \in G_{\Lambda_2}^v$  is an eigenvalue of operator  $\tilde{H}_{2\Lambda_2}^8$  if and only if it is a zero of the function  $D_{\Lambda_2}^v(z)$ , i.e.,  $D_{\Lambda_2}^v(z_0) = 0$ .

**Theorem 27.** a). If  $v=1$  and  $U < 0$ , then the operator  $\tilde{H}_{2\Lambda_2}^8$  has a unique eigenvalue  $z_2 = A - 2\sqrt{U^2 + B^2}$ , lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$ .

b). If  $v = 1$  and  $U > 0$ , then the operator  $\tilde{H}_{2\Lambda_2}^8$  has a unique eigenvalue  $\tilde{z}_2 = A + 2\sqrt{U^2 + B^2}$ , lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$ .

In the two-dimensional case, we have the analogous results.

We now consider the three-dimensional case.

**Theorem 28.** a). If  $v = 3$ ,  $U < 0$  and  $U < -\frac{3B}{W}$ , then the operator  $\tilde{H}_{2\Lambda_2}^8$  has a unique eigenvalue  $z_2$ , lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$ .

b). If  $v = 3$ ,  $U < 0$  and  $-\frac{3B}{W} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_2}^8$  has no eigenvalues, lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$ .

c). If  $v = 3$ ,  $U > 0$  and  $U > \frac{3B}{W}$ , then the operator  $\tilde{H}_{2\Lambda_2}^8$  has a unique eigenvalue  $\tilde{z}_2$ , lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$ .

d). If  $v = 3$ ,  $U > 0$  and  $0 < U \leq \frac{3B}{W}$ , then the operator  $\tilde{H}_{2\Lambda_2}^8$  has no eigenvalues, lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_2}^8$ .

We now investigated the spectrum of operator  $(\tilde{H}_{2\Lambda_3}^9 f)(\theta, \eta) = \{2A + 2B \sum_{i=1}^v [\cos \theta_i + \cos \eta_i]\} f(\theta, \eta) - U \int_{T^v} f(s, \gamma + \eta - s) ds$ .

Let  $\Lambda_3 = \theta + \eta$ , and  $\Lambda_4 = \gamma + \theta$ . Then  $(\tilde{H}_{2\Lambda_3}^9 f_{\Lambda_3})(\theta) = \{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_3^i}{2} \cos(\frac{\Lambda_3^i}{2} - \theta_i)\} f_{\Lambda_3}(\theta) - U \int_{T^v} f_{\Lambda_3}(s) ds$ .

It is known that the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_3}^9$  does not depend on the parameter  $U$  and consists of the intervals  $\sigma_{cont}(\tilde{H}_{2\Lambda_3}^9) = G_{\Lambda_3}^v = [2A - 4B \sum_{i=1}^v \cos \frac{\Lambda_3^i}{2}, 2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_3^i}{2}]$ .

We set  $D_{\Lambda_3}^v(z) = 1 - U \int_{T^v} \frac{ds_1 ds_2 \dots ds_v}{2A + 4B \sum_{i=1}^v \cos \frac{\Lambda_3^i}{2} \cos(\frac{\Lambda_3^i}{2} - s_i) - z}$ .

**Lemma 7.** The number  $z_0 \in G_{\Lambda_3}^v$  is an eigenvalue of operator  $\tilde{H}_{2\Lambda_3}^9$  if and only if it is a zero of the function  $D_{\Lambda_3}^v(z)$ , i.e.,  $D_{\Lambda_3}^v(z_0) = 0$ .

**Theorem 29.** a). If  $v = 1$  and  $U > 0$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has a unique eigenvalue  $z_3 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_3}{2}}$ , lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

b). If  $v = 1$  and  $U < 0$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has a unique eigenvalue  $\tilde{z}_3 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_3}{2}}$ , lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

In the two-dimensional case, we have the analogous results.

We now consider the three-dimensional case.

**Theorem 30.** a). If  $v = 3$ ,  $U > 0$ , and  $U > \frac{4B}{m}$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has a unique eigenvalue  $z_3$ , lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

b). If  $v = 3$ ,  $U > 0$ , and  $0 < U \leq \frac{4B}{m}$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has no eigenvalues, lying the below of continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

c). If  $v = 3$ ,  $U > 0$ , and  $U < -\frac{4B}{M}$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has a unique eigenvalue  $\tilde{z}_3$ , lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

d). If  $\nu = 3$ ,  $U < 0$ , and  $-\frac{4B}{M} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has no eigenvalues, lying the above of continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

We now using the obtaining results and representation (19), we can describe the structure of essential spectrum and discrete spectrum of the operator of third five-electron quartet state:

**Theorem 31.** If  $\nu = 1$  and  $U < 0$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_{3/2}^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^3\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + \tilde{z}_3, b + z_2 + \tilde{z}_3] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [c + \tilde{z}_1 + \tilde{z}_3, d + \tilde{z}_1 + \tilde{z}_3] \cup [e + \tilde{z}_1 + z_2, f + \tilde{z}_1 + z_2]$ , and discrete spectrum of operator  ${}^3\tilde{H}_{3/2}^q$  is consists of no more one point:  $\sigma_{disc} \left( {}^3\tilde{H}_{3/2}^q \right) = \{\tilde{z}_1 + z_2 + \tilde{z}_3\}$ , or  $\sigma_{disc} \left( {}^3\tilde{H}_{3/2}^q \right) = \emptyset$ .

Here and hereafter  $a = -2A - 4B\cos\frac{\Lambda_1}{2}$ ,  $b = -2A + 4B\cos\frac{\Lambda_1}{2}$ ,  $c = A - 2B$ ,  $d = A + 2B$ ,  $e = 2A - 4B\cos\frac{\Lambda_3}{2}$ ,  $f = 2A + 4B\cos\frac{\Lambda_3}{2}$ ,  $\tilde{z}_1 = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$ ,  $z_2 = A - 2\sqrt{U^2 + B^2}$ ,  $\tilde{z}_3 = 2A + \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_3}{2}}$ .

**Theorem 32.** If  $\nu = 1$  and  $U > 0$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_{3/2}^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^3\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + \tilde{z}_2, b + f + \tilde{z}_2] \cup [a + \tilde{z}_2 + z_3, b + \tilde{z}_2 + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + z_1 + z_3, d + z_1 + z_3] \cup [e + z_1 + \tilde{z}_2, f + z_1 + \tilde{z}_2]$ , and discrete spectrum of operator  ${}^3\tilde{H}_{3/2}^q$  is consists of no more one point:  $\sigma_{disc} \left( {}^3\tilde{H}_{3/2}^q \right) = \{z_1 + \tilde{z}_2 + z_3\}$ , or  $\sigma_{disc} \left( {}^3\tilde{H}_{3/2}^q \right) = \emptyset$ .

Here  $z_1 = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$ ,  $\tilde{z}_2 = A + 2\sqrt{U^2 + B^2}$ ,  $z_3 = 2A - \sqrt{U^2 + 16B^2\cos^2\frac{\Lambda_3}{2}}$ .

In the two-dimensional case we have the analogous results.

We now consider the three-dimensional case.

**Theorem 33.** a). If  $\nu = 3$  and  $U < 0$ ,  $U < -\frac{4B}{m}$ ,  $M > m$ ,  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $m > M$ ,  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $U < -\frac{4B}{M}$ ,  $m > M$ ,  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $M > m$ ,  $m > \frac{4}{3}W$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $m > M$ ,  $M > \frac{4}{3}W$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_{3/2}^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^3\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3] \cup [a + e + z_2, b + f + z_2] \cup [a + z_2 + \tilde{z}_3, b + z_2 + \tilde{z}_3] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [c + \tilde{z}_1 + \tilde{z}_3, d + \tilde{z}_1 + \tilde{z}_3] \cup [e + \tilde{z}_1 + z_2, f + \tilde{z}_1 + z_2]$ , and discrete spectrum of operator  ${}^3\tilde{H}_{3/2}^q$  is consists of no more one point:  $\sigma_{disc} \left( {}^3\tilde{H}_{3/2}^q \right) = \{\tilde{z}_1 + z_2 + \tilde{z}_3\}$ , or  $\sigma_{disc} \left( {}^3\tilde{H}_{3/2}^q \right) = \emptyset$ .

Here,  $a = -2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_i}{2}$ ,  $b = -2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_i}{2}$ ,  $c = A - 6B$ ,  $d = A + 6B$ ,  $e = 2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_i}{2}$ ,  $f = 2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_i}{2}$ , and  $\tilde{z}_1$ ,  $z_2$  and  $\tilde{z}_3$ , are the eigenvalues of the operators  $\tilde{H}_{2\Lambda_1}^7$ ,  $\tilde{H}_{2\Lambda_2}^8$  and  $\tilde{H}_{2\Lambda_3}^9$ , correspondingly.

b). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{3B}{W}$ ,  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{4B}{M}$ ,  $M > m$ , and  $M < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{m}$ ,  $M > m$ , and  $m > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{M}$ ,  $m > m$ , and  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{3B}{W}$ ,  $M < m$ , and  $m > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{4B}{m}$ ,  $M < m$ , and  $m < \frac{4}{3}W$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_{3/2}^q$  is consists of the union of four segments:  $\sigma_{ess} \left( {}^3\tilde{H}_{3/2}^q \right) = [a + c + e, b + d + f] \cup [a + e + z_2, b + f + z_2] \cup [c + e + \tilde{z}_1, d + f + \tilde{z}_1] \cup [c + \tilde{z}_1 + \tilde{z}_3, d + \tilde{z}_1 + \tilde{z}_3] \cup [e + \tilde{z}_1 + z_2, f + \tilde{z}_1 + z_2]$ , or

$$\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + z_2, b + f + z_2] \cup [c + e + \tilde{z}_3, d + f + \tilde{z}_3] \\ \cup [e + z_2 + \tilde{z}_3, f + z_2 + \tilde{z}_3],$$

or  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + \tilde{z}_1, b + f + \tilde{z}_1] \cup [c + e + \tilde{z}_3, d + f + \tilde{z}_3] \cup [c + \tilde{z}_1 + \tilde{z}_3, d + z_1 + z_3] \cup [e + z_1 + z_3, f + z_1 + z_3]$ , and discrete spectrum of operator  $3H3/2q$  is empty:  $\sigma_{\text{disc}} 3H32q = \emptyset$ .

c). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{3B}{W}$ ,  $M > m$ , and  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{3B}{W}$ ,  $M < m$ , and  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < -\frac{3B}{W}$ ,  $m < M$ , and  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{3B}{m}$ ,  $m < M$ , and  $m > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < -\frac{3B}{m}$ ,  $m > M$ , and  $M > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U < -\frac{4B}{m}$ ,  $M < m$ ,  $m > \frac{4}{3}W$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_{3/2}^q$  is consists of the union of two segments:  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_2, b + d + z_2]$ , or  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_3, b + d + \tilde{z}_3]$ , or  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + \tilde{z}_1, b + d + z_1]$ , and discrete spectrum of operator  $3H3/2q$  is empty:  $\sigma_{\text{disc}} 3H32q = \emptyset$ .

d). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{3B}{W} \leq U < 0$  and  $M < m$ , ( $M > m$ ),  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < 0$ ,  $M > m$ , and  $m < \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{M} \leq U < 0$ ,  $M > m$ , and  $m > \frac{4}{3}W$ , or  $U < 0$ ,  $-\frac{4B}{m} \leq U < 0$ ,  $M < m$ , and  $m > \frac{4}{3}W$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_{3/2}^q$  is single segment:  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f]$ , and discrete spectrum of operator  ${}^3\tilde{H}_{3/2}^q$  is empty:  $\sigma_{\text{disc}} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

**Theorem 34.** a). If  $\nu = 3$  and  $U > 0$ ,  $U > \frac{3B}{W}$ ,  $m > \frac{4}{3}W$ , and  $m < M$ , ( $m > M$ ), or  $U > 0$ ,  $U > \frac{4B}{m}$ ,  $m < \frac{4}{3}W$ , and  $m < M$ , or  $U > 0$ ,  $U > \frac{4B}{M}$ ,  $M < \frac{4}{3}W$ , and  $M < m$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of seven segments:

$$\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + \tilde{z}_2, b + f + \tilde{z}_2] \cup [a + \tilde{z}_2 + z_3, b + \tilde{z}_2 + z_3] \cup [c + e + z_1, d + f + z_1] \cup [c + z_1 + z_3, d + z_1 + z_3] \cup [e + z_1 + z_3, f + z_1 + z_3],$$

and discrete spectrum of operator  $3H32q$  is consists of no more one point:  $\sigma_{\text{disc}} \left( {}^3\tilde{H}_3^q \right) = \{z_1 + \tilde{z}_2 + z_3\}$ , or  $\sigma_{\text{disc}} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

Here,  $z_1$ ,  $\tilde{z}_2$  and  $z_3$ , are the eigenvalues of the operators  $\tilde{H}_{2A_1}^7$ ,  $\tilde{H}_{2A_2}^8$  and  $\tilde{H}_{2A_3}^9$ , correspondingly.

b). If  $\nu = 3$  and  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{4B}{m}$ , and  $m < M$ ,  $m < \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{m}$ ,  $M > m$ , and  $m < \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{M}$ ,  $M < m$ , and  $M < \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{4B}{M}$ , and  $M < m$ ,  $m < \frac{4}{3}W$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{3B}{W}$ , and  $m < M$ , ( $M < m$ ),  $m > \frac{4}{3}W$ , the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of four segments:  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + z_1, d + f + z_1] \cup [c + z_1 + z_3, d + z_1 + z_3]$ , or  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3] \cup [a + e + z_2, b + f + z_2] \cup [c + z_2 + z_3, d + z_2 + z_3]$ , or  $\sigma_{\text{ess}} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + \tilde{z}_2, d + f + \tilde{z}_2] \cup [c + e + z_1, d + f + z_1] \cup [e + z_1 + \tilde{z}_2, f + z_1 + z_2]$ , and discrete spectrum of operator  $3H32q$  is empty:  $\sigma_{\text{disc}} 3H32q = \emptyset$ .

c). If  $\nu = 3$  and  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{m}$ , and  $m < \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U < \frac{4B}{M}$ , and  $M < \frac{4}{3}W$ ,  $m < M$ , or  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{3B}{W}$ , and  $m < \frac{4}{3}W$ ,  $M > m$ , or  $U > 0$ ,  $\frac{4B}{M} \leq U < \frac{4B}{m}$ , and  $m > \frac{4}{3}W$ ,  $m < M$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{3B}{W}$ , and  $M < \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $\frac{4B}{m} \leq U < \frac{4B}{M}$ , and  $M > \frac{4}{3}W$ ,  $M < m$ , then the essential



spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of two segments:  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + c + z_3, b + d + z_3]$ , or  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [a + e + \tilde{z}_2, b + f + \tilde{z}_2]$ , or  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f] \cup [c + e + z_1, d + f + z_1]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

d). If  $\nu = 3$  and  $U > 0$ ,  $0 < U \leq \frac{3B}{W}$ , and  $M < \frac{4}{3}W$ , or  $U > 0$ ,  $0 < U \leq \frac{3B}{W}$ , and  $m < \frac{4}{3}W$ ,  $M < m$ , or  $U > 0$ ,  $0 < U \leq \frac{4B}{M}$ , and  $M > \frac{4}{3}W$ ,  $M > m$ , or  $U > 0$ ,  $0 < U \leq \frac{4B}{M}$ ,  $M > m$ , and  $m > \frac{4}{3}W$ , or  $U > 0$ ,  $0 < U \leq \frac{4B}{m}$ ,  $M < m$ , and  $m > \frac{4}{3}W$ , or  $U > 0$ ,  $0 < U \leq \frac{4B}{m}$ ,  $M < m$ , and  $M > \frac{4}{3}W$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is single segment:  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a + c + e, b + d + f]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

We now consider the three-dimensional case, when  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ . Then the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_1}^7$  is consists of the segment  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_1}^7 \right) = G_{\Lambda_1}^3 = \left[ -2A - 12B\cos\frac{\Lambda_1^0}{2}, -2A + 12B\cos\frac{\Lambda_1^0}{2} \right]$ .

**Theorem 35.** a). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U < 0$ , and  $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_1}^7$  has only one eigenvalue  $z_1^1$ , lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

b). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_1}^7$  has no eigenvalue, lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

c). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U > 0$ , and  $U > \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_1}^7$  has only one eigenvalue  $z_1^2$ , lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

d). If  $\nu = 3$  and  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $U > 0$ , and  $0 < U \leq \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_1}^7$  has no eigenvalue, lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_1}^7$ .

We now consider the three-dimensional case, when  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ , and  $\Lambda_4 = (\Lambda_4^0, \Lambda_4^0, \Lambda_4^0)$ . Then the continuous spectrum of the operator  $\tilde{H}_{2\Lambda_3}^9$  is consists of the segment  $\sigma_{cont} \left( \tilde{H}_{2\Lambda_3}^9 \right) = G_{\Lambda_3}^3 = \left[ 2A - 12B\cos\frac{\Lambda_3^0}{2}, 2A + 12B\cos\frac{\Lambda_3^0}{2} \right]$ .

**Theorem 36.** a). If  $\nu = 3$  and  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ , and  $\Lambda_4 = (\Lambda_4^0, \Lambda_4^0, \Lambda_4^0)$ , and  $U < 0$ , and  $U < -\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has only one eigenvalue  $z_3^1$ , lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

b). If  $\nu = 3$  and  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ , and  $\Lambda_4 = (\Lambda_4^0, \Lambda_4^0, \Lambda_4^0)$ , and  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W} \leq U < 0$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has no eigenvalue, lying the above of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

c). If  $\nu = 3$  and  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ , and  $\Lambda_4 = (\Lambda_4^0, \Lambda_4^0, \Lambda_4^0)$ , and  $U > 0$ , and  $U > \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has only one eigenvalue  $z_3^2$ , lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

d). If  $\nu = 3$  and  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ , and  $\Lambda_4 = (\Lambda_4^0, \Lambda_4^0, \Lambda_4^0)$ , and  $U > 0$ , and  $0 < U \leq \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$ , then the operator  $\tilde{H}_{2\Lambda_3}^9$  has no eigenvalue, lying the below of the continuous spectrum of operator  $\tilde{H}_{2\Lambda_3}^9$ .

We now using the obtaining results and representation (19), we can describe the structure of essential spectrum and discrete spectrum of the operator of third five-electron quartet state:

Let  $\Lambda_1 = (\Lambda_1^0, \Lambda_1^0, \Lambda_1^0)$ , and  $\Lambda_3 = (\Lambda_3^0, \Lambda_3^0, \Lambda_3^0)$ , and  $\Lambda_4 = (\Lambda_4^0, \Lambda_4^0, \Lambda_4^0)$ .

Theorem 37. a). If  $\nu = 3$ , and  $U < 0$ ,  $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $U < -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of seven segments:  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [a_1 + z_2^1 + z_3, b_1 + z_2^1 + z_3] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [c_1 + z_1^1 + z_3, d_1 + z_1^1 + z_3] \cup [e_1 + z_1^1 + z_3, f_1 + z_1^1 + z_3]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc}\left({}^3\tilde{H}_3^q\right) = \{z_1^1 + z_2^1 + z_3\}$ , or  $\sigma_{disc}\left({}^3\tilde{H}_3^q\right) = \emptyset$ .

Here, hereafter  $a_1 = 2A - 12B\cos\frac{\Lambda_1^0}{2}$ ,  $b_1 = 2A + 12B\cos\frac{\Lambda_1^0}{2}$ ,

$c_1 = 2A - 12B\cos\frac{\Lambda_2^0}{2}$ ,  $d_1 = 2A + 12B\cos\frac{\Lambda_2^0}{2}$ ,  $e_1 = A - 6B$ ,  $f_1 = A + 6B$ , and  $z_1^1$ ,  $z_2^1$ , and  $z_3$ , are the eigenvalues of the operators  $\tilde{H}_{2\Lambda_1}^7$ ,  $\tilde{H}_{2\Lambda_2}^8$  and  $\tilde{H}_{2\Lambda_3}^9$ , correspondingly.

b). If  $\nu = 3$ , and  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{3B}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{3B}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ ,  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of four segments:  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + z_2^1 + z_3, b_1 + z_2^1 + z_3]$ , or  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [c_1 + z_1^1 + z_3, d_1 + z_1^1 + z_3]$ , or  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1] \cup [c_1 + z_1^1 + z_3, d_1 + z_1^1 + z_3]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^3\tilde{H}_3^q\right) = \emptyset$ .

c). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{3B}{W} \leq U \leq -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} < \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_2^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ , and  $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \leq U \leq -\frac{3B}{W}$ ,  $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$ , and  $\cos\frac{\Lambda_1^0}{2} > \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of two segments:  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [c_1 + e_1 + z_1^1, d_1 + f_1 + z_1^1]$ , or  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^1, b_1 + f_1 + z_2^1]$ , or  $\sigma_{ess}\left({}^3\tilde{H}_3^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc}\left({}^3\tilde{H}_3^q\right) = \emptyset$ .

d). If  $\nu = 3$  and  $U < 0$ ,  $-\frac{3B}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $-\frac{3B}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , or  $U < 0$ ,  $-\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U < 0$ , and  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is single segment:  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

**Theorem 38.** a). If  $\nu = 3$ , and  $U > 0$ ,  $U > \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $U > \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $U > \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $U > \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of seven segments:  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_{22}, b_1 + f_1 + z_{22} \cup a_1 + z_{22} + z_3, b_1 + z_{22} + z_3] \cup [c_1 + e_1 + z_{12}, d_1 + f_1 + z_{12}] \cup [c_1 + z_{12} + z_3, d_1 + z_{12} + z_3] \cup [e_1 + z_{12} + z_2, f_1 + z_{12} + z_2]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is consists of no more one point:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \{z_1^2 + z_2^2 + z_3\}$ , or  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

Here,  $z_1^2$ ,  $z_2^2$ , and  $z_3$ , are the eigenvalues of the operators  $\tilde{H}_{2A_1}^7$ ,  $\tilde{H}_{2A_2}^8$  and  $\tilde{H}_{2A_3}^9$ , correspondingly.

b). If  $\nu = 3$ , and  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ , and  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ ,  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of four segments:  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [c_1 + e_1 + z_{12}, d_1 + f_1 + z_{12}] \cup [c_1 + z_{12} + z_3, d_1 + z_{12} + z_3]$ , or  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [a_1 + z_2^2 + z_3, b_1 + z_2^2 + z_3]$ , or  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2] \cup [e_1 + z_1^2 + z_2^2, f_1 + z_1^2 + z_2^2]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

c). If  $\nu = 3$  and  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{3B}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U \leq \frac{12B\cos \frac{A_2^0}{2}}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U < \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} < \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $\frac{12B\cos \frac{A_2^0}{2}}{W} \leq U < \frac{3B}{W}$ ,  $\cos \frac{A_1^0}{2} > \cos \frac{A_2^0}{2}$ , and  $\cos \frac{A_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is consists of the union of two segments:

$\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [c_1 + e_1 + z_1^2, d_1 + f_1 + z_1^2]$ , or  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 +$

$d_1 + f_1] \cup [a_1 + e_1 + z_2^2, b_1 + f_1 + z_2^2]$ , or  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3, b_1 + d_1 + z_3]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

d). If  $\nu = 3$  and  $U > 0$ ,  $0 < U < \frac{3B}{W}$ , and  $\cos \frac{\Lambda_1^0}{2} < \cos \frac{\Lambda_2^0}{2}$ ,  $\cos \frac{\Lambda_1^0}{2} > \frac{1}{4}$ , or  $0 < U < \frac{3B}{W}$ , and  $\cos \frac{\Lambda_1^0}{2} > \cos \frac{\Lambda_2^0}{2}$ ,  $\cos \frac{\Lambda_2^0}{2} > \frac{1}{4}$ , or  $U > 0$ ,  $0 < U < \frac{12B\cos \frac{\Lambda_2^0}{2}}{W}$ , and  $\cos \frac{\Lambda_1^0}{2} < \cos \frac{\Lambda_2^0}{2}$ ,  $\cos \frac{\Lambda_1^0}{2} < \frac{1}{4}$ , or  $U > 0$ ,  $0 < U < \frac{12B\cos \frac{\Lambda_2^0}{2}}{W}$ , and  $\cos \frac{\Lambda_1^0}{2} > \cos \frac{\Lambda_2^0}{2}$ ,  $\cos \frac{\Lambda_2^0}{2} < \frac{1}{4}$ , then the essential spectrum of the third five-electron quartet state operator  ${}^3\tilde{H}_3^q$  is single segment:  $\sigma_{ess} \left( {}^3\tilde{H}_3^q \right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$ , and discrete spectrum of operator  ${}^3\tilde{H}_3^q$  is empty:  $\sigma_{disc} \left( {}^3\tilde{H}_3^q \right) = \emptyset$ .

## 6. Conclusion

In this paper we consider five-electron systems in the sextet and first, and second, and third quartet states. In the five-electron systems the total spin  $S$  take the values  $S = \frac{5}{2}, \frac{3}{2}$ , and  $\frac{1}{2}$ . The states with total spin value  $S = \frac{5}{2}$ , so-called the sextet state. We proved in the sextet state the spectrum of the system purely continuous and consists of the segment  $[m_\nu, M_\nu] = [5A - 4B\nu, 5A + 4B\nu]$ , and in the system five-electron bound states or five-electron anti-bound states are absent. The state with total spin value  $S = \frac{3}{2}$ , so-called the quartet states, in the system exists four type quartet states. In the first five-electron quartet state, corresponds the basic function the form  ${}^1q_{m,n,r,t,l \in Z^\nu}^{3/2} = a_{m,\downarrow}^+ a_{n,\uparrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ . We proved, in the case, when  $\nu=1$ , the essential spectrum of the first five-electron quartet state operator is consists of the union of seven segments, and the discrete spectrum of the first five-electron quartet state operator is consists of no more one point. In the system exists no more one five-electron bound states or no more one five-electron anti-bound states. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the first five-electron quartet state operator is consists or the union of seven segments, or the union of four segments, or of the union of two segments, or of single segments, and the discrete spectrum of first five-electron quartet state operator is consists no more one point. Consequently, in this case the system have no more one five-electron bound states or no more one five-electron anti-bound states. In the second five-electron quartet state, corresponds the basic function the form  ${}^2q_{m,n,r,t,l \in Z^\nu}^{3/2} = a_{m,\uparrow}^+ a_{n,\downarrow}^+ a_{r,\uparrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ . We prover, in one-dimensional case, the essential spectrum of the second five-electron quartet state operator is consists of the union of seven segments, and the discrete spectrum of the second five-electron quartet state operator is consists of no more one point. In the system exists no more one five-electron bound states or no more one five-electron anti-bound states. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the second five-electron quartet state operator is consists or the union of seven segments, or the union of four segments, or the union of two segments, or of single segments, and the discrete spectrum of second five-electron quartet state operator is consists no more one point. Consequently, in this case the system have no more one five-electron bound states or no more one five-electron anti-bound states. In the third five-electron quartet state, corresponds the basic function the form  ${}^3q_{m,n,r,t,l \in Z^\nu}^{3/2} = a_{m,\uparrow}^+ a_{n,\uparrow}^+ a_{r,\downarrow}^+ a_{t,\uparrow}^+ a_{l,\uparrow}^+ \varphi_0$ . If  $\nu = 1$ , then the essential spectrum of the five-electron third quartet state operator is consists of the union of seven segments, and discrete spectrum five-electron third quartet state operator is consists of no more one point. In two-dimensional case, we have the analogous results. In the three-dimensional case the essential spectrum of five-electron third quartet state operator is consists of the union of seven, or of the union of four, or of the union of two, or of single segments, and the discrete spectrum of this operator is consists of no more one point. Consequently, in this case the system have no more one five-electron bound states or no more one five-electron anti-bound states.

Comparing of theorems 8,9,10,11,13,14 with theorems 16,17,18,19,22,23, as well as with theorems 31,32,33,34,37,38, show that the spectra of these three Quartet States are the different, i.e. these three Quartet States has a different origins.

## References

- J. Hubbard. "Electron correlations in narrow energy bands." Proc. Roy. Soc. A., 1963. V. 276. P. 238-257.  
M. C. Gutzwiller. "Effect of Correlation on the Ferromagnetism of Transition Metals", Phys. Rev. Lett., 1963. V. 10. P.159-162.  
J. Kanamori. "Electron correlation and ferromagnetism of transition metals". Prog. Theor. Phys. 1963. V. 30. P. 275-289.  
P. W. Anderson. "Localized Magnetic States in Metals". Phys. Rev. 1961. V. 124. P. 41-53.

- S. P. Shubin, S. V. Wonsowsky. "On the electron theory of metals". Proc. Roy. Soc. A. 1934. V. 145. P. 159-172.
- B. V. Karpenko, V. V. Dyakin, and G. L. Budrina. "Two electrons in the Hubbard Model". Phys. Met. Metallogr. 1986. V. 61. P. 702-706.
- D. Mattis. "The few-body problems on a lattice". Rev. Mod. Phys. 1986. V. 58. P. 370-379.
- E. Lieb. "Two theorems on the Hubbard Model". Phys. Rev. Lett., 1989. V. 62. P. 1201-1204.
- A. M. Tsvelick, and P. B. Wiegman. "Exact results in the theory of magnetic alloys". Adv. Phys. 1983. V. 32 (4). P. 453-713.
- Yu. A. Izyumov and Yu. N. Skryabin. "Statistical Mechanics of Magnetically Ordered Systems", [in Russian]. Nauka, Moscow. 1987; English transl. Consultants Bureau. New York. 1988.
- S. M. Tashpulatov. "Spectral Properties of three-electron systems in the Hubbard Model". Theoretical and Mathematical Physics., 2014. V. 179 (3). P. 712-728.
- S. M. Tashpulatov. "Spectra of the energy operator of four-electron systems in the triplet state in the Hubbard Model". Journal Phys. Conf. Ser. 2016. V. 697. 012025, P. 1-25.
- S. M. Tashpulatov. "The structure of essential spectra and discrete spectrum of four-electron systems in the Hubbard model in a singlet state." Lobachevskii Journal of Mathematics. 2017. V. 38 (3). P. 530-541.
- M. Reed, and B. Simon. Methods of Modern Mathematical Physics, Vol. 1, Functional Analysis. Acad. Press, New York. 1972.
- M. Reed, and B. Simon. Methods of Modern Mathematical Physics, Vol. 4, Operator Analysis. Acad. Press, New York. 1982.
- T. Ichinose. Spectral Properties of Tensor Products of Linear Operators.1. Trans. American Math.Soc. 1978. Vol. 235. P. 75-113.
- T. Ichinose. Spectral Properties of Tensor Products of Linear Operators.2: The Approximate Point Spectrum and Kato Essential Spectrum. Trans. American Math. Soc. 1978. Vol. 237. P. 223-254.
- T. Ichinose. Tensor Products of Linear Operators. Spectral Theory. Banach Center Publications.PWN-Polish Scientific Publishers, Warsaw. 1982. Vol. 8. P. 294-300.
- M. A. Neimark. Normed Rings.[in Russian], Nauka, Moscow. 1968. English transl. prev. ed., Noordhoff, Groningen, 1959.
- V. V. Valkov, and S. G. Ovchinnikov, and O. P. Petrakovskii. "The Excitation Spectra of two-magnon systems in easy-axis quasidimensional Ferromagnetism". Sov. Phys. Solid State. 1986. V. 30. P. 3044-3047.