American Review of Mathematics and Statistics
June 2021, Vol. 9, No. 1, pp. 1-11
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v9n1a1
URL: https://doi.org/10.15640/arms.v9n1a1

Fredholm Properties for Pencils

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Abstract

This paper is to interest the Fredholm properties of the operator Pencil. In particular, we detect and approximate the spectra of Fredholm operator Pencils via Green's kernel (contour integral) with consider Exponential solutions of differential equations with operator coefficients. A motive for this article is to gain a deeper understanding the development of aspects of the theory of ordinary differential equations with operator coefficients by concentrating on some specific examples of trapped modes. The results of this paper, it is established directly that Fredholm operator Pencil and the index calculated without the need to consider adjoint operator. Also, we leverage some concepts to go from the semi-Fredholm property to the Fredholm property.

Keywords: Fredholm operator, Operator Pencils, Green Kernel, Sobolev spaces, Projection of Pencils, semi-Fredholm property.

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1 General Background

1.1 Spaces and Operators

A polynomial operator Pencil also called *operator polynomial*, which is an expression of the form

$$\mathcal{B}_A(\mu) = \mu^n A_0 + \mu^{n-1} A_1 + ... + A_n, \tag{1}$$

where A_j $(j = 0, 1, ..., n)$ is operator acting in a Hilbert space H and $\mu \in \mathbb{C}$ is the spectral parameter.

This is a linear subspace of the Hilbert space H with the norm given by

$$\|u\|^2_{H_k} = \sum_{j=0}^{\infty} (1 + \lambda_j^2)^{\frac{k}{2}} |a_j|^2,$$

for $u \in H_k$ and operator $\mathcal{B}_A$ will be introduced below: An operator Pencil

$$\mathcal{B}_A : \mathbb{C} \to B(H_2, H_0),$$

which is defined as,

$$\mathcal{B}_A(\mu) = \mu^2 + A - \lambda, \tag{2}$$

In order to define the function space on which A given in (2) and values of parameters α and β, which are related to approximate eigenvalues of an operator Pencil $\mathcal{B}_A$ we need to introduce the exponential weighted function spaces modelled on Sobolev spaces to examine the operator Pencil. These spaces defined by the following finite norm:

$$\|u\|^2_{W^k_{\alpha,\beta}} := \sum_{j=0}^{k} \int_{-\infty}^{0} e^{2\alpha t} \left\| D_t^j u \right\|^2_{H_{k-j}} dt + \sum_{j=0}^{k} \int_{0}^{\infty} e^{2\beta t} \left\| D_t^j u \right\|^2_{H_{k-j}} dt;$$

where $W^k_{\alpha,\beta}$ denotes the set of $u : \mathbb{R} \to H_k$, for $k \in \mathbb{N}_0$, the operator $D_t = -i\dfrac{d}{dt}$ on $\mathbb{R}$, and $\alpha, \beta \in \mathbb{R}$. (See [5], [6], [7]).

1.2 Fredholm Operator Pencil

We define basic facts of the Fredholm operator Pencil and its adjoint; these are collected without proof. Then, we can structure of the formula of $\mathcal{B}_A^{-1}(\mu)$ near the pole.

Definition 1.1. We can consider the operator Pencil $\mathcal{B}_A$ such that

$$\mathcal{B}_A : \mathbb{C} \to B(H_2, H_0)$$

$$\mathcal{B}_A(\mu) = \mu^2 + A - \lambda \quad \text{for} \quad \mu \in \mathbb{C}.$$

is called Fredholm for all $\mu \in \mathbb{C}$, and it is invertible at least one value of μ (see, for example, [9] and [10]).

Theorem 1.1. Let Ω be in the domain $\mathbb{C}$. Suppose the operator Pencil $\mathcal{B}_A(\mu)$ satisfies the following conditions:

1) $\mathcal{B}_A(\mu) \in \Phi(H_2, H_0)$ for all $\mu \in \Omega$.

2) There exists a number $\mu \in \Omega$ such that the operator $\mathcal{B}_A(\mu)$ has a bounded inverse.

Then, the spectrum of operator pencil $\mathcal{B}_A(\mu)$ consists of isoloted eigenvalues with finite algebraic multiplicity. See, for example, [9] and [10].

In what follows, we consider the operator Pencil again and the definition of adjoint operator.

Definition 1.2. The adjoint operator Pencil $\mathcal{B}_A^* : \mathbb{C} \to B(H_0^*, H_2^*)$ is a Fredholm operator for all $\overline{\mu} \in \mathbb{C}$ and invertible at least one value and therefore its spectrum is discrete. See [9].

Proposition 1.2. Let $\mathcal{B}_A$ be a Fredholm operator Pencils. Then,

- $\mu_0 \in \mathbb{C}$ is an eigenvalue of $\mathcal{B}_A$ if and only if $\overline{\mu_0}$ is an eigenvalue of $\mathcal{B}_A^*$.
- The geometric and algebraic multiplicity of μ and $\overline{\mu}$ coincide.

Proof. The reader can see the proof of this proposition in [9] and [10]. □

The main purpose in the following part is defined the inverse operator $\mathcal{B}_A^{-1}$ of operator pencil $\mathcal{B}_A$ near an eigenvalue μ_0, we need the notion of Holomorphic function. Then, we consider some properties of this operator which will be used to investigate some arguments of this thesis.

Definition 1.3. Let Ω be a domain in Complex plane $\mathbb{C}$. An operator function

$$\Upsilon(\mu) : \Omega \to B(H_2, H_0)$$

is called Holomorphic on Ω when it can be represented as a power series

$$\Upsilon(\mu)=\sum_{j=0}^{\infty}\Upsilon_j(\mu-\mu_0)^j,\quad \Upsilon_j\in B(H_2,H_0),$$

which is convergent in $B(H_2,H_0)$ in a neighbourhood of $\mu_0\in\Omega$ (see [9]).

Theorem 1.3. Let μ_0 be an eigenvalue of $\mathcal{B}_A$ and let J and $m_1,...,m_J$ be its geometric multiplicity and partial multiplicity respectively. Suppose that

$$\{\varphi_{k,s}\},\quad s=0,...,m_k-1,\quad k=1,...,J$$

is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0.

(i) There exits a unique

$$\{\psi_{k,s}\},\quad s=0,...,m_k-1,\quad k=1,...,J$$

is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$.

Such that in a neighbourhood of μ_0, the resolvent operator (inverse operator) can be represented as

$$\mathcal{B}_A^{-1}(\mu)=\sum_{k=1}^{J}\sum_{h=0}^{m_k-1}\frac{P_{k,h}}{(\mu-\mu_0)^{m_k-h}}+\Upsilon(\mu),\tag{3}$$

where,

$$P_{k,h}=\sum_{s=0}^{h}\langle .,\psi_{k,s}\rangle_{H_0}\varphi_{k,h-s},\tag{4}$$

and Υ is a Holomorphic function in the neighbourhood of μ_0.

(ii) The system $\{\psi_{k,s}\}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ satisfies the bi-orthogonal condition that is,

$$\sum_{s=0}^{d}\sum_{n=s+1}^{m_k+s}\frac{1}{n!}(\mathcal{B}_A^{(n)}(\mu_0)\varphi_{k,m_k+s-n},\psi_{j,d-s})_{H_0}=\delta_k\delta_d\tag{5}$$

for $k,\ j=1,...,J$, and $d=0,...,m_k-1$.

(iii) Suppose $\psi_{j,0},...,\psi_{j,m_j-1}$ for $j=1,...,J$ is a collection of Jordan chain of $\mathcal{B}^*(A)$ corresponding to $\overline{\mu_0}$ which is subject to (5), then the collection $\psi_{j,0},...,\psi_{j,m_j-1}$ is a canonical system satisfying (i).

Proof. The reader can see the proof in [9] and [10]. □

2 Green's Kernel

What is a Green's function? Mathematically, it is the kernel of an integral operator that represent the inverse of a differential operator (see [4]). In this section, we construct bases to define the Green's function with some properties.

The following assertion will use to define a Green's function of the resolvent operator and the bounded map $\mathcal{B}_A(D_t) : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$.

Lemma 2.1. Suppose $\alpha \notin \Gamma(\mathcal{B}_A) = \Im(\sigma(\mathcal{B}_A))$, that is the line $\Im(\sigma(\mathcal{B}_A))$ does not contain eigenvalues of the operator Pencils $\mathcal{B}_A(\mu)$.

Then the Green's function is defined by

$$G(t) = \frac{1}{2\pi} \int_{\Im\mu=\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu.$$

However, the following proposition, we observe the integral of the inverse operator is convergent in the norm of $B(H_0, H_2)$ to determine $G(t)$.

Proposition 2.2. For $\alpha \notin \Gamma(\mathcal{B}_A)$, i.e., the line $\Im(\sigma(\mathcal{B}_A))$ does not contain eigenvalues of the operator pencils $\mathcal{B}_A(\mu)$. Then for $t \neq 0$, the limit

$$\lim_{R\to\infty} \int_{-R+i\alpha}^{R+i\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu,$$

exists in $B(H_0, H_2)$.

Remark 1. From above arguments, we have the operator

$$G(t) = \frac{1}{2\pi} \int_{\Im\mu=\alpha} e^{it\mu} \mathcal{B}_A^{-1}(\mu) d\mu.$$

explained in sense of the Cauchy integral. To get that,

$$G(t) = -\frac{1}{2\pi t} \int_{\Im\mu=\alpha} e^{it\mu} D_\mu \mathcal{B}_A^{-1}(\mu) d\mu$$

with absolute convergent in $B(H_0, H_2)$.

2.1 Representations for G(t)

Now, we observe the difference between $G(t)$ and $G^{(\beta)}(t)$.

$G(t)$ does not depend on α, we can set $\Sigma_{\alpha\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lessgtr \alpha\}$ and we consider $\Im\mu = \beta$.

A new Green's kernel is defined by

$$G^{(\beta)}(t) = \frac{1}{2\pi}\int_{\Im\mu=\beta} e^{it\mu}\mathcal{B}_A^{-1}(\mu)d\mu.$$

To understand this relation between $G(t)$ and $G^{(\beta)}(t)$, we have the following theorems.

Theorem 2.3. Let the operator is defined by

$$P_v(t) = \frac{1}{2\pi}\int_{S_v} e^{it(\mu-\mu_v)}\mathcal{B}_A^{-1}(\mu)d\mu,$$

where S_v is a small circle centred at the eigenvalue μ_v. Then, we have that

$$P_v(t) = i\sum_{k=1}^{J}\sum_{h=0}^{m_k-1}\frac{(it)^h}{h!}P_{k,h}, \tag{6}$$

where, $P_{k,h}$ is defined in (4), i.e.,

$$P_{k,h} = \sum_{s=0}^{h}\langle ., \psi_{k,s}\rangle_{H_0}\varphi_{k,h-s}, \tag{7}$$

and J be a geometric multiplicity of μ_0.

Theorem 2.4. Suppose there are no eigenvalues of the operator Pencil $\mathcal{B}_A$ on the lines $\Im\mu = \beta$, and $\Sigma_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lessgtr \alpha\}$. Then

$$G(t) = \sum_{\mu\in\Sigma_{\alpha_+}} e^{i\mu t}P_v(t) + G^{(\beta)}(t), \tag{8}$$

$$G(t) = -\sum_{\mu\in\Sigma_{\alpha_-}} e^{i\mu t}P_v(t) + G^{(\beta)}(t). \tag{9}$$

Therefore, the formula (8) and (9) are the new representation of $G(t)$ as $t \to \pm\infty$. See for example, [9] and [11].

Theorem 2.5. For $k = 1, 2, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all $\mu \in \Sigma_{\alpha_\pm}$ then we have

$$\sum_{h=0}^{m_k-1}\frac{(it - i\tau)^h}{h!}P_{k,m_k-1-s} = \sum_{h=0}^{m_k-1}\langle ., D_\tau\Psi(it)\psi_k\rangle_{H_0}D_t\Phi_k(it),$$

Theorem 2.6. For $k = 1, 2, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all μ then the Green's kernel has new representation

$$G(t) - G^{(\alpha)}(t) = -i \sum_{\mu\in\Sigma_{\alpha_+}} \sum_{k=0}^{J} \sum_{h=0}^{m_k-1} e^{i\mu_0 t}\langle ., \psi_{k,s}\rangle_{H_0}\varphi_{k,h-s}.$$

We can consider the function $G^{(\beta)}(t)$ by the following lemma:

Lemma 2.7. For $\alpha \notin \Gamma(\mathcal{B}_A)$, and we set $\Sigma_{\alpha_\pm} = \{\mu \in \sigma(\mathcal{B}_A) : \Im\mu \lessgtr \alpha\}$. Then,

$$G^{(\beta)}(t) = \begin{cases} i\sum_{\mu\in\Sigma_{\alpha_+}} Res(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & \text{if} \quad t > 0 \\ -i\sum_{\mu\in\Sigma_{\alpha_-}} Res(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & \text{if} \quad t < 0. \end{cases}$$

The following Lemma, we can generalise the new representation of $G^{\beta}(t)$.

Lemma 2.8. Suppose $\alpha, \beta \in \mathbb{R}\backslash\Gamma(\mathcal{B}_A)$, and We have note that $\Sigma_{\beta_+} \subseteq \Sigma_{\alpha_+}$, $\Sigma_{\alpha_-} \subseteq \Sigma_{\beta_-}$, and $\Sigma_{\alpha,\beta} = \Sigma_{\alpha_+}\backslash\Sigma_{\beta_+} = \Sigma_{\beta_-} \setminus \Sigma_{\alpha_-}$. Then,

$$G^{(\beta)}(t) - G^{(\alpha)}(t) = \begin{cases} i\sum_{\mu\in\Sigma_{\beta_+}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) - i\sum_{\mu\in\Sigma_{\alpha_+}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & t > 0 \\ -i\sum_{\mu\in\Sigma_{\beta_-}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) + i\sum_{\mu\in\Sigma_{\alpha_-}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) & t < 0, \end{cases}$$

It follows,

$$G^{(\beta)}(t) - G^{(\alpha)}(t) = -i \sum_{\mu\in\Sigma_{\alpha,\beta}} \mathrm{Res}(e^{it\mu}\mathcal{B}_A^{-1}(\mu); \mu) \quad \text{for all} \quad t. \tag{10}$$

Lemma 2.9. If

$$\mathcal{B}_A^{-1}(\mu) = \langle ., \psi_{k,h-s}\rangle_{H_0}\varphi_{k,s}(\mu - \mu_0)^{h-m_k} + \Upsilon(\mu)$$

for μ is neighbourhood of μ_0. Then,

$$\mathcal{B}_A(\mu)\varphi_{k,s} = 0.$$

Corollary 2.10. Similarly, the adjoint of $\mathcal{B}_A$ we have that

$$\mathcal{B}_A^*(\overline{\mu})\psi_{k,s} = 0.$$

We have Theorem 2.6 achieves to find the solution for the difference two solutions of non-homogeneous equation

$$\mathcal{B}_A(D_t)u = f. \tag{11}$$

We have $\alpha \leq \beta$ and $\Sigma_{\alpha,\beta}$ denote the linear span of the set of all exponential solutions corresponding to $\mu_0 \in \sigma(\mathcal{B}_A)$.

Then, we have the following propositions:

Proposition 2.11. Let $\alpha \leq \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$ and suppose $f \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$. Choose the unique $u_\alpha \in W^2_{\alpha,\alpha}$ and $u_\beta \in W^2_{\beta,\beta}$ such that

$$\mathcal{B}_A(D_t)u_\alpha = f \quad \text{and} \quad \mathcal{B}_A(D_t)u_\beta = f.$$

Then, the difference $u_\alpha - u_\beta$ lies in $\Sigma_{\alpha,\beta}$ (see, for example, [2] and [9]).

Proposition 2.12. For $\alpha, \beta \in \mathbb{R} \setminus \Gamma$, and we have the maps

$$A^{(\alpha)} = D_t^2 + A - \lambda : W^2_{\alpha,\alpha} \longrightarrow W^0_{\alpha,\alpha},$$

$$A^{(\beta)} = D_t^2 + A - \lambda : W^2_{\beta,\beta} \longrightarrow W^0_{\beta,\beta},$$

are isomorphisms.

Let $f \in W^0_{\alpha,\alpha} \cap W^0_{\beta,\beta}$, and $u_\alpha \in W^2_{\alpha,\alpha}$, $u_\beta \in W^2_{\beta,\beta}$ be the solutions of

$$A^{(\alpha)}u_\alpha = f \quad \text{and} \quad A^{(\beta)}u_\beta = f,$$

respectively. Then,

$$u_\alpha(t) - u_\beta(t) = \sum_{\mu \in \Sigma\alpha,\beta} \int_{\mathbb{R}} e^{i\mu_0(t-s)} P_{k,h} f(s) ds.$$

Proof. By Lemma 2.1, and Theorem 2.6 we can observe directly,

$$\begin{aligned} u_\alpha(t) - u_\beta(t) &= \int_{\mathbb{R}} G^{(\alpha)}(t-s)f(s)ds - \int_{\mathbb{R}} G^{(\beta)}(t-s)f(s)ds \\ &= \int_{\mathbb{R}} (G^{(\alpha)} - G^{(\beta)})(t-s)f(s)ds \\ &= \sum_{\mu \in \Sigma\alpha,\beta} \sum_{h=0}^{m_k-1} \int_{\mathbb{R}} e^{i\mu_0(t-s)} P_{k,h} f(s) ds. \end{aligned}$$

Remark 2. For $\{\varphi_{k,s}\}_{s=0}^{m_k-1}$ is a canonical system of Jordan of $\mathcal{B}_A$ corresponding to μ_0, and $\{\psi_{k,h-s}\}_{s=0}^{m_k-1}$ is a canonical system of Jordan of $\mathcal{B}_A^*$ corresponding to $\overline{\mu_0}$ for $k = 1, ..., J$ and $s = 0, ..., m_k - 1$, and these conditions hold for all $\mu \in \Sigma_{\alpha,\beta}$, by Theorem 2.6, we can set

$$u_\mu(t) = -ie^{i\mu_0 t}\varphi_{k,s},$$

and

$$v_\mu(t) = e^{i\overline{\mu_0} t}\psi_{k,h-s},$$

for $k = 1, ..., J$ and $s = 0, ..., m_k - 1$.

Thus, u_μ and v_μ are called exponential solutions of $\mathcal{B}_A(D_t)u_\mu = 0$, and $\mathcal{B}_A^*(D_t)v_\mu = 0$, respectively, (see [9], pp. $10 - 11$).

Proposition 2.13. We have $u_\mu(t) = -ie^{i\mu_0 t}\varphi_{k,h-s}$, and $v_\mu(t) = e^{i\overline{\mu_0} t}\psi_{k,s}$, and by using Proposition 2.12, we can get

$$\begin{aligned}(A^{(\beta)})^{-1}f - (A^{(\alpha)})^{-1}f &= \int_{\mathbb{R}} (G^{(\beta)} - G^{(\alpha)})(t-s)f(s)ds \\ &= \sum_{\mu\in\Sigma\alpha,\beta} \sum_{h=0}^{m_k-1} \int_{\mathbb{R}} (-ie^{i\mu_0 t}\varphi_{k,h-s})\langle e^{i\overline{\mu_0} s}\psi_{k,s}, f(s)\rangle_{H_0} ds \\ &= \sum_{\mu\in\Sigma\alpha,\beta} u_\mu \langle v_\mu, f\rangle_{H_0}.\end{aligned}$$

3 Main Results

Theorem 3.1. Let $\alpha, \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$. Then, $\mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta}$ is semi-Fredholm with a finite-dimensional kernel.

Theorem 3.2. Let $\beta \in \mathbb{R}$. Then, the map $A^{(\beta)} = \mathcal{B}_A(D_t) : W^0_{\beta,\beta} \to W^0_{\beta,\beta}$ has a finite-dimensional kernel.

Proposition 3.3. For $\alpha \leq \beta$, we can consider the maps

$$A^{(\alpha,\beta)} = \mathcal{B}_A(D_t) : W^2_{\alpha,\beta} \to W^0_{\alpha,\beta},$$

and

$$A^{(\beta,\alpha)} = \mathcal{B}_A(D_t) : W^2_{\beta,\alpha} \to W^0_{\beta,\alpha},$$

and we have that solutions $\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}$ and $\{v_\mu : \mu \in \Sigma_{\alpha,\beta}\}$ of the equations $\mathcal{B}_A(D_t)u_\mu = 0$ and $\mathcal{B}_A(D_t)v_\mu = 0$, respectively, and are linearly independent sets.

To observe the following claims:

- Claim (i):

$$\operatorname{Ker} A^{(\alpha,\beta)} = \{u \in W^2_{\alpha,\beta} : A^{(\alpha,\beta)}u = 0\} = \{0\}.$$

- Claim (ii):

$$\operatorname{Ran} A^{(\alpha,\beta)} = \{f \in W^0_{\alpha,\beta} : \langle v_\mu, f\rangle = 0 \quad \text{for all} \quad \mu \in \Sigma_{\alpha,\beta}\}.$$

- Claim (iii):

$$\operatorname{Ker} A^{(\beta,\alpha)} = \operatorname{Span}\{u_\mu : \mu \in \Sigma_{\alpha,\beta}\}.$$

- Claim (iv): We have

$$\operatorname{Ran} A^{(\beta,\alpha)} = W^0_{\beta,\alpha}.$$

Corollary 3.4. Let $\alpha, \beta \in \mathbb{R} \setminus \Gamma(\mathcal{B}_A)$, Suppose

$$A^{(\alpha)} : W^2_{\alpha,\alpha} \to W^0_{\alpha,\alpha}$$

and

$$A^{(\beta)} : W^2_{\beta,\beta} \to W^0_{\beta,\beta}$$

are isomorphism maps. Then, $A^{(\alpha)}$ and $A^{(\beta)}$ are Fredholm maps with index 0.

We finish this part, by the last result in the current thesis which shows how the index of the Fredholm maps $A^{(\alpha,\beta)}$ and $A^{(\beta,\alpha)}$ varies when we change α and β.

Theorem 3.5. Suppose $\alpha < \beta \in \mathbb{R}\backslash\Gamma$. Then the maps

$$A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta}$$

and

$$A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}$$

are Fredholm maps with

$$\text{Index } A^{(\alpha,\beta)} : W^2_{\alpha,\beta} \longrightarrow W^0_{\alpha,\beta} = -|\Sigma_{\alpha,\beta}| = -\text{ Index } A^{(\beta,\alpha)} : W^2_{\beta,\alpha} \longrightarrow W^0_{\beta,\alpha}.$$

Acknowledgement

I would like to express my sincerest thanks to Dr. Daniel Elton for the fruitful discussion forms, and his comments on some results to help me to improve it. I would also like to thank the anonymous referee for helping to moral support.

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