American Review of Mathematics and Statistics
December 2019, Vol. 7, No. 2, pp. 3-9
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v7n2a2
URL: https://doi.org/10.15640/arms.v7n2a2

Quenching Time for Some Semilinear Equations with A Potential

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Abstract.

This paper concerns the study of a semilinear parabolic equation subject to Neumann boundary conditions, with a potential and positive initial datum. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. Finally, we give some numerical results to illustrate our analysis.

Key-words: *Quenching, semilinear parabolic equation, numerical quenching time.*

AMS subject classification (2010): *35B40, 35B50, 35K60, 65M06.*

1-Introduction

Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = \Delta u - c(x,t)u^{-p(x)} \; in \; \Omega \times (0,\mathrm{T}), \quad (1)$$

$$\frac{\partial u}{\partial \nu} = 0 \; on \; \partial\Omega \times (0,\mathrm{T}), \quad (2)$$

$$\mathrm{u}(\mathrm{x},0) = \mathrm{u}_0(\mathrm{x}) > 0 \; in \; \bar{\Omega}, \quad (3)$$

where Δ is the Laplacian, ν the exterior normal unit vector on $\partial\Omega$. We suppose that the initial datum

$$\mathrm{u}_0 \in \mathrm{C}^2(\bar{\Omega}) \text{ and } \mathrm{u}_0(\mathrm{x}) > 0 \; in \; \bar{\Omega}.$$

Here the potential $c(x,t)$ is a nonnegative locally Hölder continuous function defined for $x \in \bar{\Omega}$ and $t \geq 0$. The exponent $\mathrm{p} \in \mathrm{C}^0(\Omega)$, $0 < p_0 = inf_{x\in\bar{\Omega}} p(x) < sup_{x\in\bar{\Omega}} p(x) = p_+$. Here (0, T) is the maximal time interval of existence of the solution u, and by a solution, we mean the following.

Definition1.1. *A solution of (1)-(3) is a function* $u(x,t)$ *continuous in* $\bar{\Omega} \times [0,T)$, $u(x,t) > 0 \; in \; \bar{\Omega} \times [0,T)$, *and twice continuously differentiable in* x *and once in* t *in* $\Omega \times (0,T)$.

The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally.
When T is finite, then the solution u develops a quenching in a finite time, namely $\lim_{t\to T} \mathrm{u}_{min}(\mathrm{t}) = 0$, where $\mathrm{u}_{min}(\mathrm{t}) = \min_{x\in\bar{\Omega}} \mathrm{u}(\mathrm{x},\mathrm{t})$. In this last case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u.

Since the pioneering work of Kawarada in 1975 (see, [25]), the study of the phenomenon of quenching for semilinear heat equations has attracted a considerable attention (see, for example [3]-[4], [6]-[8], [11], [14], [24], [26], [28]-[30], [36] and the references cited therein). More precisely, in [7] Boni has studied the problem (1)-(3) for the phenomenon of blow-up. He has given some sufficient conditions under which solutions to such equation tend to zero or blow up in a finite time. In the same way, some authors have proved the existence and uniqueness of solution (see, [16], [27]). In [8], Boni and Kouakou have treated a similar problem with variable exponent. They have estimated the quenching time and studied its continuity as a function of the initial datum u_0. The originality of this work is that it is the first attempt of studying the phenomenon of quenching with variable exponent and a potential depending both on space and time. Using standard methods, the local in time existence and uniqueness of solutions can be easily proved (see, [7], [16]).

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Our aim in this paper consists in showing that, under some hypotheses, the solution of (1)-(3) quenches in a finite time. If we set $g(x,u) = c(x,t)u^{-p(x)}$, then we observe that the function g is continuous in both variables and locally Lipschitz in the second one. Let us notice that, because the initial datum of the problem considered is sufficiently regular, the solution of this problem exists and is regular. In addition, we note that the regularity of solution is as important as the regularity of the initial data, and the maximum principle holds (see, [16], [27], [33]). This paper is structured as follows. In the following section, we show that under some assumptions, the solution u of (1)-(3) quenches in a finite time and estimate its quenching time and finally, in the last section, we give some numerical results to illustrate our analysis.

2- Quenching time

In this section, using an idea of Friedman and McLeod in [17], we may prove the following result on the quenching of the solution u of (1)-(3).

Theorem 2.1. *Suppose that there exists a constant* $A \in (0, 1]$ *such that the initial datum at (3) satisfies*

$$\Delta u_0(x) - c(x,t)(u_0(x))^{-p(x)} \leq -Ac(x,t)(u_0(x))^{-p_0} \quad in\ \ \Omega. \tag{4}$$

Then, the solution u *of (1)-(3) quenches in a finite time* T *which obeys the following estimate*

$T \leq \frac{(u_{0min})^{p_0+1}}{AM(p_0+1)}$,

where M is some positive constant.

Proof. We know that (0, T) is the maximal time interval of existence of the solution u. Therefore, to prove our theorem, we have to show that T is finite and satisfies the above inequality. For this fact, we introduce $J(x,t)$ a function defined as follows

$$J(x,t) = u_t(x,t) + Ac(x,t)\big(u(x,t)\big)^{-p_0} \text{ in } \overline{\Omega} \times [0,T).$$

The derivative of J in t yields $J_t = u_{tt} - p_0 Ac(x,t)u^{-p_0-1}u_t$ and by a simple calculation we obtain

$$J_t - \Delta J = (u_t - \Delta u)_t - Ap_0 c(x,t)u^{-p_0-1}u_t - Ac(x,t)\Delta u^{-p_0} \quad in\ \ \Omega \times (0,T). \tag{5}$$

It is not hard to see that $\Delta u^{-p_0} = p_0(p_0+1)u^{-p_0-2}|\nabla u|^2 - p_0 u^{-p_0-1}\Delta u$ in $\Omega \times (0,T)$, which implies that $\Delta u^{-p_0} \geq -p_0 u^{-p_0-1}\Delta u$ in $\Omega \times (0, T)$. Applying this inequality in (5), we find that

$$J_t - \Delta J \leq (u_t - \Delta u)_t - Ap_0 c(x,t)u^{-p_0-1}(u_t - \Delta u) \ in\ \Omega \times (0,T). \tag{6}$$

Use (1) and (6) to obtain

$$J_t - \Delta J \leq c(x,t)p(x)u^{-p(x)-1}u_t + Ap_0(c(x,t))^2 u^{-p_0-1-p(x)} \ in\ \Omega \times (0,T).$$

Due to the fact that $p_0 \leq p(x)$ in Ω, we discover that

$$J_t - \Delta J \leq c(x,t)p(x)u^{-p(x)-1}(u_t + Ac(x,t)u^{-p_0}) \ in\ \Omega \times (0,T).$$

Making use of the expression of J, we derive the following inequality

$$J_t - \Delta J \leq c(x,t)p(x)u^{-p(x)-1}J \ \ in\ \ \Omega \times (0,T).$$

The boundary condition (2) allows us to write

$$\frac{\partial J}{\partial \nu} = \left(\frac{\partial u}{\partial \nu}\right)_t - Ap_0 c(x,t)u^{-p_0-1}\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T).$$

According to (4), we have

$$J(x,0) = \Delta u_0(x) - c(x,t)\big(u_0(x)\big)^{-p(x)} + Ac(x,t)\big(u_0(x)\big)^{-p_0} \leq 0 \ \ in\ \ \Omega.$$

One concludes by the maximum principle that $J(x,t) \leq 0$ in $\Omega \times (0, T)$, that is

$$u_t(x,t) + Ac(x,t)\big(u(x,t)\big)^{-p_0} \leq 0 \ in\ \ \Omega \times (0,T). \tag{7}$$

By the definition of $c(x,t)$, we have $c(x,t) \leq M$ where M is some positive constant.

Thus, the estimate (7) may be rewritten as follows

$$u^{p_0}du \leq -AMdt \quad in\ \ \Omega \times (0,T). \tag{8}$$

Integrate the above inequality over (0, T) to obtain

$$T \leq \frac{\big(u(x,0)\big)^{p_0+1}}{AM(p_0+1)} \ \text{ for } x \in \Omega.$$

We deduce that

$$T \leq \frac{(u_{0min})^{p_0+1}}{AM(p_0+1)}.$$

We observe that the quantity on the right-hand side of the above inequality is finite. Consequently, u quenches at the time T and the proof is finished. □

3-Numerical results

To compute the numerical results, we need to consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \Delta u - c(x,t)u^{-p(x)} \ \ in \ \ B \times (0,T),$$
$$\frac{\partial u}{\partial \nu} = 0 \ on \ \ S \times (0,T),$$
$$u(x,0) = u_0(x) \ \ in \ B,$$

where $c(x,t) = C(\|x\|,t), p(x) = \psi(\|x\|), u_0(x) = \varphi(\|x\|), B = \{x\epsilon R^N; \|x\| < 1\}, S = \{x\epsilon R^N; \|x\| = 1\}$. Another form of the above problem is

$$u_t = u_{rr} + \frac{N-1}{r}u_r - C(r,t)u^{-\psi(r)}, r \in (0,1), t \in (0,T) \tag{9}$$

$$u_r(0,t) = 0, u_r(1,t) = 0, \qquad t \in (0,T), \tag{10}$$

$$u(r,0) = \phi(\mathrm{r}), \mathrm{r} \in [0,1], \tag{11}$$

where, we take $C(r,t) = \frac{r+1}{t+1}$, $\psi(\mathrm{r}) = 1 + \frac{\varepsilon r}{r+1}$ with $\varepsilon \epsilon [0,1]$ and $\phi(\mathrm{r}) = 4 + 3\cos(\pi \mathrm{r})$. In order to compute the numerical solution, we need to construct an adaptive scheme. For this fact, define the grid $x_i = ih, \ 0 \leq i \leq I$, where I is a positive integer and $\mathrm{h} = 1/\mathrm{I}$. Approximate the solution u of (9)-(11) by the solution $U_h^{(n)} = \left(U_0^{(n)}, \cdots, U_I^{(n)}\right)^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - C_0^{(n)}\left(U_0^{(n)}\right)^{-\psi_0},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} - C_i^{(n)}\left(U_i^{(n)}\right)^{-\psi_i}, 1 \leq i \leq I-1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{U_{I-1}^{(n)} - U_I^{(n)}}{h^2} + (N-1)\frac{U_I^{(n)} - U_{I-1}^{(n)}}{2h} - C_I^{(n)}\left(U_I^{(n)}\right)^{-\psi_I},$$

$$U_i^{(0)} = \varphi_i, \ \ 0 \leq i \leq I,$$

Where $C_i^{(n)} = \frac{ih+1}{t_n+1}$, $\psi_i = 1 + \frac{\varepsilon ih}{ih+1}$ and $\varphi_i = 4 + 3\cos(\pi ih)$. For the time step we take

$$\Delta t_n = min\left\{\frac{(1-h^2)h^2}{2N}, h^2(U_{hmin}^{(n)})^{p_+ + 1}\right\}$$

with $U_{hmin}^{(n)} = min_{0 \leq i \leq I} U_i^{(n)}$.

This condition permits to the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T and ensures the positivity of the discrete solution. An important fact concerning the phenomenon of quenching is that, if the solution u quenches at the time T, then, when the time t approaches the quenching time T, the solution u decreases to zero rapidly. We also approximate the solution u of (9)-(11) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta\, t_n}=N\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2}-C_0^{(n)}\left(U_0^{(n)}\right)^{-\psi_0-1}U_0^{(n+1)},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta\, t_n}=\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h}-C_i^{(n)}\left(U_i^{(n)}\right)^{-\psi_i-1}U_i^{(n+1)},$$
$$1\le i\le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta\, t_n}=\frac{U_{I-1}^{(n)}-U_I^{(n)}}{h^2}+(N-1)\frac{U_I^{(n)}-U_{I-1}^{(n)}}{2h}-C_I^{(n)}\left(U_I^{(n)}\right)^{-\psi_I-1}U_I^{(n+1)},$$

$$U_i^{(0)}=\varphi_i,\quad 0\le i\le I.$$

As in the case of the explicit scheme, here again, we transform our scheme to an adaptive one by choosing $\Delta t_n=h^2(U_{hmin}^{(n)})^{p_+ +1}$.

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution is also guaranteed using standard methods (see for instance [6]). It is not hard to see that

$$u_{rr}(0,t)=lim_{r\to 0}\frac{u_r(r,t)}{r}.$$

On the other hand, according to (10), we have

$$\frac{u_r(r,t)}{r}=0.$$

Hence, if $r=0$ and $r=1$, we obtain

$$u_t(0,t)=Nu_{rr}(0,t)-C(0,t)u^{-\psi(0)}(0,t),\qquad t\in(0,T),$$
$$u_t(1,t)=Nu_{rr}(1,t)-C(1,t)u^{-\psi(1)}(1,t),\qquad t\in(0,T).$$

These observations have been taken into account in the construction of our schemes when $i=0$ and $i=I$.
We need the following definition.

Definition 3.1. *We say that the discrete solution* $U_h^{(n)}$ *of the explicit scheme or implicit scheme quenches in a finite time if* $lim_{n\to\infty}U_{hmin}^{(n)}=0$ *and the series* $\sum_{n=0}^{\infty}\Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty}\Delta t_n$ *is called the numerical quenching time of the discrete solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time $T^n=\sum_{j=0}^{n-1}\Delta t_j$ which is computed at the first time when

$$\Delta t_n=|T^{n+1}-T^n|\le 10^{-16}.$$

The order(s) of the method is computed from

$$s=\frac{\log\left(\frac{T_{4h}-T_{2h}}{T_{2h}-T_h}\right)}{\log(2)}.$$

Numerical experiments for $\psi_i=1+\frac{\varepsilon i h}{ih+1},\ N=2$

First case: ε = 0

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	2,094707	3975	-	-
32	2,188877	16064	1	-
64	2,239645	64921	5	0.89
128	2,266009	252408	51	0.94
256	2,279445	992631	3180	0.97

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t_n	n	CPU time	s
16	2,094457	3974	1	-
32	2,188816	16064	2	-
64	2,239630	63920	16	0.89
128	2,266006	252408	242	0.94
256	2,279444	992630	7620	0.98

Second case: ε = 1/10

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	2,135560	3973	-	-
32	2,233026	16080	1	-
64	2,285571	64050	7	0.89
128	2,312858	253140	62	0.94
256	2,326764	996283	3012	0.97

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t_n	n	CPU time	s
16	2,135296	3973	-	-
32	2,232961	16079	2	-
64	2,285554	64050	17	0.89
128	2,312854	253140	275	0.94
256	2,326763	996283	7740	0.97

Third case: ε = 1/1000

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	2,095107	3975	-	-
32	2,189310	16064	2	-
64	2,240095	63922	7	0.89
128	2,266468	252414	53	0.94
256	2,279908	992663	4560	1.08

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t_n	n	CPU time	s
16	2,094857	3974	-	-
32	2,189248	16064	2	-
64	2,240080	63921	22	0.89
128	2,266451	252414	256	0.94
256	2,279908	992663	2743	0.97

Remark3.1. *If we consider the problem (9)-(11) in the case where the potential* $C(r,t) = \frac{r+1}{t+1}$, *the exponent of the nonlinear source* $\Psi(r) = 1 + \frac{\varepsilon r}{1+r}$ *with* $\varepsilon = 0$, *and the initial datum* $\phi(r) = 4 + 3\cos(\pi r)$, *we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is slightly equal to that in which the exponent of the nonlinear source increases slightly, that is when* ε *is a small positive real (see, Tables 1-6 for an illustration). This result confirms the theory established in the previous section.*

In what follows, we give some plots to illustrate our analysis. In Figures 1 and 2, we can appreciate that the discrete solution quenches in a finite time. We also remark that the representation of the discrete solution when ε=0 is practically the same that the one when ε=1/10.

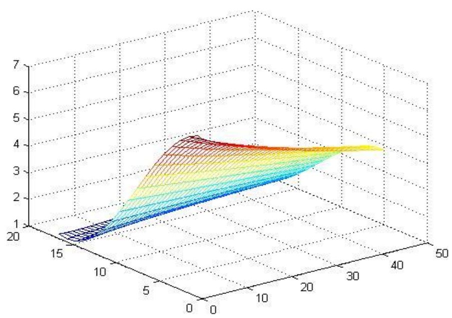

Figure 1: Evolution of the explicit discrete solution: ε= 0

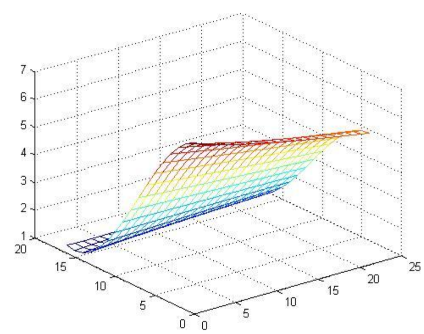

Figure 2: Evolution of the implicit discrete solution: ε = 1/10

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