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Applying Euler's Formula to Integrate $e^{a\theta}sin(b\theta)$ and $e^{a\theta}cos(b\theta)$

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Abstract

The dreaded "integration by parts" can be seen as a calculus student's worst nightmare, since it can often cause difficulties to new students. Students commonly miss the significance of understanding imaginary numbers, since these topics are often overlooked or even skipped in most algebra and calculus courses.. In this article, I examine an efficient method to integrate functions of the forms: $e^{a\theta}sin(b\theta)$and $e^{a\theta}cos(b\theta)$ by using Euler's formula and complex numbers with the goal of giving students an alternative to integrating by parts for functions of these forms.

Keywords: Integration by Parts, Euler's Formula, Techniques of Integration, Mathematics Education.

The purpose of this short article is to show a connection and a contrast between common practices in a calculus classroom (ie: integration by parts) and another unorthodox approach to integrating functions of these particular forms. The following is a very practical lesson because it is a useful way to develop student insight across mathematical topics, such as the significance of Euler's formula, while using the complex number system to solve integration problems. When I presented this technique to local calculus teachers at a conference, it had an extremely positive reception, as it is not the "usual" approach to integrating functions of this form. Many members of the audience commented on the ease of the technique as well as the time savings while integrating functions of these forms. Since the mission of this journal strives to strengthen connections between research and practice, this small article is a good fit. Furthermore, the convenient application of imaginary numbers serves as a motivating example to expand the students' overall knowledge of complex numbers.

Below, I offer a very efficient method, using imaginary numbers, to integrate functions of the forms: $e^{a\theta}sin(b\theta)$and $e^{a\theta}cos(b\theta)$. Functions of these forms present themselves quite often in engineering applications of differential equations, notably, when implementing Laplace transforms. Frequently, the complex number system is quickly skimmed over in the classroom, and some valuable applications are overlooked. For instance, "integration by parts" is often a topic that students struggle with, in particular when the problem requires more than one application of the technique, and students may miss the opportunity to see the significance of understanding imaginary numbers. The following method offers an alternative to integrating by parts, and it gives a gratifying application of the utilization of imaginary numbers.

From a pedagogical perspective, I find that the lesson is best taught after techniques for both integrating exponential functions and integrating by parts have been covered. When taught in this order, the students tend to appreciate the method using Euler's formula more and regard it as a "shortcut". Developing critical thinking skills is also one of the overarching goals of the lesson. Before beginning the lesson, a brief discussion on the value of a complex number $(a + bi)$ is recommended, in particular, that $(a + bi)$is, and should be treated as, a constant number with two parts.

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Integration using Euler's formula:

Note: Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$**, or :** $e^{ib\theta} = \cos b\theta + i\sin b\theta$

The first step is just algebraic manipulation and the implementation of Euler's formula. Direct the students to apply Euler's formula to put the expression $\int e^{(a+bi)\theta}\,d\theta$ into the form $(u+vi)$, as shown below.

(1) $$\int e^{(a+bi)\theta}d\theta = \int e^{a\theta}e^{ib\theta}d\theta = \int e^{a\theta}(\cos b\theta + i\sin b\theta)d\theta = \int (e^{a\theta}\cos b\theta)d\theta + i\int (e^{a\theta}\sin b\theta)d\theta$$

This step gives students an opportunity to develop critical thinking skills in order to arrive at this form. It typically requires approximately ten minutes, while the instructor resists the urge to give them the final form. Note that in this first step no integration actually takes place, and the resulting expression has a real component and an imaginary component. These are both points of discussion in the classroom.

The following step is integration of a simple exponential function and then a subsequent implementation of Euler's formula. The students can then rewrite the denominator as a real number by multiplying the numerator and the denominator by the conjugate of the denominator.

(2) $$\int e^{(a+bi)\theta}d\theta = \frac{e^{(a+bi)\theta}}{a+bi} = \frac{e^{a\theta}(\cos b\theta + i\sin b\theta)}{a+bi} = \frac{e^{a\theta}(\cos b\theta + i\sin b\theta)(a-bi)}{a^2+b^2}$$

$$= \frac{e^{a\theta}(a\cos b\theta + b\sin b\theta)}{a^2+b^2} + i\frac{e^{a\theta}(a\sin b\theta - b\cos b\theta)}{a^2+b^2}$$

Once the like terms are combined, the students see that the resulting expression has both real and imaginary parts. At this point, two different representations have been found for the same expression, and each of these representations have both real and imaginary parts. These two representations of $\int e^{(a+bi)\theta}d\theta$ lead to the integration for $\int (e^{a\theta}\cos b\theta)d\theta$ and $\int (e^{a\theta}\sin b\theta)d\theta$. The instructor should be able to prod the students into discovering how. That is, simply equate the real parts and imaginary parts of the expressions in (1) and (2).

$$\boxed{\int (e^{a\theta}\cos b\theta)d\theta = \frac{e^{a\theta}(a\cos b\theta + b\sin b\theta)}{a^2+b^2}} \qquad \boxed{\int (e^{a\theta}\sin b\theta)d\theta = \frac{e^{a\theta}(a\sin b\theta - b\cos b\theta)}{a^2+b^2}}$$

In the above calculations, the benefits of using imaginary numbers and Euler's theorem are profound. The amount of work is cut in more than half by applying this method instead of the traditional integration by parts. Most importantly, the students will gain an appreciation for using imaginary numbers in certain applications. I know this to be a valuable lesson, because often students will ask afterwards whether there are other such "shortcuts" using imaginary numbers, and they begin to see some applications to using imaginary numbers, such as calculating and developing formulas for derivatives to trigonometric functions. This lesson segues well into a discussion of De Moivre's formula, i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. I have found that the students' appreciation of using imaginary numbers greatly improves, and they are far less likely to come to an incorrect answer by using this technique to integrate functions of these forms.

References

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