American Review of Mathematics and Statistics
June 2017, Vol. 5, No. 1, pp. 58-67
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v5n1a6
URL: https://doi.org/10.15640/arms.v5n1a6

Normal CR-Submanifolds of a Quasi Kaehlerian Manifold

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Abstract:

In this paper, we establish a mathematical identity, which makes it possible to use the Gauss formula and Weingarten formula in the anti invariant distribution.Then we give some sufficient and necessary conditions for normal CR-submanifold of a quasi Kaehlerian manifold by both tensor S and S^* of type (1, 2).

Keywords: quasi Kaehlerian manifold, CR-submanifold, normal, connection

1 Introduction

In this paper, all manifolds and morphisms are supposed to be differentiable of class C^∞. Let $\overline{M}$ be a real n-diemensional connected differentiable manifold. $T(\overline{M})$ and $F(\overline{M})$ are respectively the tangle bundle to $\overline{M}$ and the algebra of differentiable functions on $\overline{M}$. Also, we denote by $\Gamma(H)$ the module of differentiable sections of a vector bundle H.

A linear connection on $\overline{M}$ is a mapping

$\overline{\nabla}$: $\Gamma(T\overline{M})\times\Gamma(T\overline{M})\to\Gamma(T\overline{M})$; $(X, Y)\to\overline{\nabla}_X Y$

satisfying the following conditions

(1) $\overline{\nabla}_{f(X)+Y}(Z) = f\overline{\nabla}_X Z+\overline{\nabla}_Y Z$,

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(2) $\overline{\nabla}_X(fY+Z)=f\overline{\nabla}_X Y+(Xf)Y+\overline{\nabla}_X Z$, for any $f\in F(\overline{M})$ and $X, Y, Z\in\Gamma(T\overline{M})$. The operetor $\overline{\nabla}_X$ is called the covariant differentiation with respect to X . Thus for any tensor field Θ of type (0, s) or (1, s) we define the covariant differentiation $\overline{\nabla}_X\Theta$ of Θ with respect to X by $(\overline{\nabla}_X\Theta)(X_1, X_2,\ldots, X_s)=\overline{\nabla}_X(\Theta(X_1, X_2,\ldots, X_s))-\sum_{i=1}^{s}\Theta(X_1,\ldots,\overline{\nabla}_X X_i,\ldots, X_s)$, (1.1)

for any $X_i\in\Gamma(T\overline{M})$, i =1, 2, ..., s . A linear connection $\overline{\nabla}$ on $\overline{M}$ is said to be a Riemannian connection if Riemannian metric g satisfying $Xg(Y, Z)=g(\overline{\nabla}_X Y, Z)+g(Y, \overline{\nabla}_X Z)$, (1.2)

for any $X, Y\in\Gamma(T\overline{M})$. An almost complex structure on $\overline{M}$ is a tensor field J of type (1, 1) on $\overline{M}$ such that at every point $x\in\overline{M}$ we have $J^2=-I$, where I denotes the identify transformation of $T_x\overline{M}$. A manifold $\overline{M}$ endowed with an almost complex structure is called an almost complex manifold. The covariant derivative of J is defined by $(\overline{\nabla}_X J)Y=\overline{\nabla}_X JY-J\overline{\nabla}_X Y$, (1.3)

for any $X, Y\in\Gamma(T\overline{M})$. More, we define the torsion tensor of J or the Nijenhuis tensor of J by

$$[J, J](X, Y)=[JX, JY]-[X, Y]-J[JX, Y]-J[X, JY], \qquad (1.4)$$

for any $X, Y\in\Gamma(T\overline{M})$, where $[X, Y]$ is the Lie bracket of vector fields X and Y , that is, $[X, Y]=\overline{\nabla}_X Y-\overline{\nabla}_Y X$. A Hermitian metric on an almost complex manifold $\overline{M}$ is a Riemannian metric g satisfying $g(JX, JY)=g(X, Y)$, (1.5)

for any $X, Y\in\Gamma(T\overline{M})$. An almost complex manifold endowed with a Hermitian metric is said to be an almost Hermitian manifold. Definition 1.1([3]). An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a quasi Kaehlerian manifold if we have $(\overline{\nabla}_X J)Y+(\overline{\nabla}_{JX} J)JY=0$, (1.6)

for any $X, Y\in\Gamma(T\overline{M})$. Definition 1.2([1]). An almost Hermitian manifold $\overline{M}$ with Levi-Civita connection $\overline{\nabla}$ is called a Kaehlerian manifold if we have $\overline{\nabla}_X J=0$, (1.7)

for any $X\in\Gamma(T\overline{M})$. Obviously, a Kaehlerian manifold is a quasi Kaehlerian manifold. Let M be an m -dimensional Riemannian submanifold of an n -dimensional Riemannian manifold $\overline{M}$. We denote by $TM^{\perp}$ the normal bundle to M and by g both metric on M and $\overline{M}$. Also, we denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M , and denote by $\nabla^{\perp}$ the induced normal connection on M .

Then, for any $X, Y\in\Gamma(TM)$ we have $\overline{\nabla}_X Y=\nabla_X Y+h(X, Y)$, (1.8) where h: $\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM^{\perp})$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.8) is called the Gauss formula and h is called the second fundamental form of M. Now, for any $X\in\Gamma(TM)$ and $V\in\Gamma(TM^{\perp})$ we denote by $-A_V X$ and $\nabla_X^{\perp}V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have $\overline{\nabla}_X V=-A_V X+\nabla_X^{\perp}V$. (1.9)

Thus, for any $V\in\Gamma(TM^{\perp})$ we have a linear operator, satisfying

$$g(A_V X, Y)=g(X, A_V Y)=g(h(X, Y), V). \qquad (1.10)$$

The equation (1.9) is called the Weingarten formula. An m-dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assignes to each point x of $\overline{M}$ an m-dimensional linear subspace D_x of $T_x\overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x\in D_x$ for each $x\in\overline{M}$. When this happens we write $X\in\Gamma(D)$. The distribution D is said to be differentiable if for any $x\in\overline{M}$ there exist m differentiable linearly independent vector fields $X_i\in\Gamma(D)$ in a neighborhood of x. From now on, all distributions are supposed to be differentiable of class C^{∞}. Definition 1.3([1]). Let $\overline{M}$ be a real n-dimensional almost Hermitian manifold with almost complex structure J and with Hermitian metric g. Let M be a real m-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. Then M is called a CR-submanifold of $\overline{M}$ if there exist a differentiable distribution D: $x\to D_x\subset T_xM$, on M satisfying the following conditions: (1) D is holomorphic, that is, $J(D_x)=D_x$, for each $x\in M$,

(2)the complementary orthogonal distribution $D^{\perp}$: $x\to D_x^{\perp}\subset T_xM$, is anti-invariant, that is, $J(D_x^{\perp})\subset T_xM^{\perp}$, for each $x\in M$. Now let M be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold $\overline{M}$. For each vector field X tangent to M, we put $JX=\phi X+\omega X$, (1.11)

where ϕX and ωX are respectively the tangent part and the normal part of JX. We denote by P and Q respectively the projection morphisms of TM to D and $D^{\perp}$, that is,

$$X=PX+QX, \qquad (1.12)$$

for any $X\in\Gamma(TM)$. Then we have

$$\phi X=JPX \qquad (1.13)$$

and

$$\omega X=JQX, \qquad (1.14)$$

for any $X\in\Gamma(TM)$. Moreover, we have

$$\phi^2=-P \qquad (1.15)$$

and

$$\phi^3 + \phi = 0. \qquad (1.16)$$

Next, for each vector field V normal to M, we put

$$JV = BV + CV, \qquad (1.17)$$

where BV and CV are respectively the tangent part and the normal part of JV.

We take account of the decomposition $T\overline{M} = D \oplus D^{\perp} \oplus JD^{\perp} \oplus \nu$. Obviously, we have $\phi X \in \Gamma(D)$, $\omega X \in \Gamma(JD^{\perp})$, $BV \in \Gamma(D^{\perp})$ and $CV \in \Gamma(\nu)$, for any $X \in \Gamma(TM)$ and $V \in \Gamma(JD^{\perp} \oplus \nu)$. Further, we obtain $B \circ \omega = -Q$.

The covariant derivative of ϕ is defined by

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \qquad (1.18)$$

for any $X, Y \in \Gamma(TM)$. On the other hand, the covariant derivative of ω is defined by

$$(\nabla_X \omega)Y = \nabla_X^{\perp} \omega Y - \omega \nabla_X Y, \qquad (1.19)$$

for any $X, Y \in \Gamma(TM)$. The exterior derivative of ω is given by

$$d\omega(X, Y) = \frac{1}{2}\{\nabla_X^{\perp} \omega Y - \nabla_Y^{\perp} \omega X - \omega([X, Y])\}, \qquad (1.20)$$

for any $X, Y \in \Gamma(TM)$.

Remark: The more details of exterior derivative is founed in [2]. The Nijenhuis tensor of ϕ is defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \qquad (1.21)$$

for any $X, Y \in \Gamma(TM)$, where $[X, Y]$ is the Lie bracket of vector fields X and Y. We define two the tensor fields S and S^* respectively by $S(X, Y) = [\phi, \phi](X, Y) - 2Bd\omega(X, Y)$, (1.22)

and $S^*(Y, X) = (L_Y \phi)X = [Y, \phi X] - \phi[Y, X]$, (1.23)

for any $X, Y \in \Gamma(TM)$. Definition 1.4([1]). The CR-submanifold M is said to be normal if

$$S(X, Y) = 0, \qquad (1.24)$$

for any $X, Y \in \Gamma(TM)$. Definition 1.5. The CR-submanifold M is said to be mixed normal if

$$S(X, Y) = 0, \qquad (1.25)$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$.

2 Main Results

Lemma 2.1. Let $\overline{M}$ be a quasi Kaehlerian manifold. Then we have

$$(\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X = \frac{1}{2}J[J, J](X, Y), \qquad (2.1)$$

for any $X, Y \in \Gamma(T\overline{M})$.

Proof: For any $X, Y \in \Gamma(T\overline{M})$. From (1.4) and (1.3) we acquire

$$[J, J](X, Y) = (\overline{\nabla}_{JX} J)Y - (\overline{\nabla}_{JY} J)X + J(\overline{\nabla}_Y J)X - J(\overline{\nabla}_X J)Y. \qquad (2.2)$$

Using (2.2), (1.6) and (1.3) we have

$$[J, J](X, Y) = (\overline{\nabla}_X J)JY - (\overline{\nabla}_Y J)JX + J(\overline{\nabla}_Y J)X - J(\overline{\nabla}_X J)Y$$

$$= 2J((\overline{\nabla}_Y J)X - (\overline{\nabla}_X J)Y). \qquad (2.3)$$

(2.3) follows that (2.1) holds. Q.E.D.

Lemma 2.2. Let $\overline{M}$ be a quasi Kaehlerian manifold. Then we have

$$(\overline{\nabla}_{JX} J)Y - (\overline{\nabla}_{JY} J)X = \frac{1}{2}[J, J](X, Y), \qquad (2.4)$$

for any $X, Y \in \Gamma(T\overline{M})$. Proof: For any $X, Y \in \Gamma(T\overline{M})$. From (1.6) we get

$$(\overline{\nabla}_{JX} J)Y - (\overline{\nabla}_{JY} J)X = -(\overline{\nabla}_{JX} J)J^2Y + (\overline{\nabla}_{JY} J)J^2X$$

$$= (\overline{\nabla}_X J)JY - (\overline{\nabla}_Y J)JX. \qquad (2.5)$$

Using (1.3) in (2.5) we obtain

$$(\overline{\nabla}_{JX} J)Y - (\overline{\nabla}_{JY} J)X = -J((\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X). \qquad (2.6)$$

(2.4) comes from (2.6). Q.E.D.

Lemma 2.3. Let $\overline{M}$ be a quasi Kaehlerian manifold. Then we have

$$(\nabla_X \phi)Y = A_{\omega Y}X + Bh(X, Y) + \nabla_{\phi X}Y + \phi\nabla_{\phi X}\phi Y$$

$$+ Bh(\phi X, \phi Y) - \phi A_{\omega Y}\phi X + B\nabla^{\perp}_{\phi X}\omega Y, \qquad (2.7)$$

$$(\nabla_X \omega)Y = -h(X, \phi Y) + Ch(X, Y) + h(\phi X, Y) + \omega\nabla_{\phi X}\phi Y$$

$$+ Ch(\phi X, \phi Y) - \omega A_{\omega Y}\phi X + C\nabla^{\perp}_{\phi X}\omega Y, \qquad (2.8)$$

for any $X \in \Gamma(D), Y \in \Gamma(TM)$.

Proof: For any $X \in \Gamma(D), Y \in \Gamma(TM)$. Using (1.6) and (1.3), we have

$$(\overline{\nabla}_X JY - J\overline{\nabla}_X Y) + (-\overline{\nabla}_{JX}Y - J\overline{\nabla}_{JX}JY) = 0. \qquad (2.9)$$

Taking into account (1.11), (2.9) becomes

$$(\overline{\nabla}_X \phi Y + \overline{\nabla}_X \omega Y) - J\overline{\nabla}_X Y - \overline{\nabla}_{\phi X}Y - J(\overline{\nabla}_{\phi X}\phi Y + \overline{\nabla}_{\phi X}\omega Y) = 0. \qquad (2.10)$$

Taking account of (1.8) and (1.9), (2.10) changes into

$$\nabla_X \phi Y + h(X, \phi Y) - A_{\omega Y} X + \nabla^{\perp}_X \omega Y - J\nabla_X Y - Jh(X, Y) - \nabla_{\phi X} Y - h(\phi X, Y)$$

$$- J\nabla_{\phi X} \phi Y - Jh(\phi X, \phi Y) + JA_{\omega Y}\phi X - J\nabla^{\perp}_{\phi X} \omega Y = 0 . \qquad (2.11)$$

According to (1.11) and (1.17), (2.11) turns into

$$\nabla_X \phi Y + h(X, \phi Y) - A_{\omega Y} X + \nabla^{\perp}_X \omega Y - \phi\nabla_X Y - \omega\nabla_X Y - Bh(X, Y) - Ch(X, Y)$$

$$- \nabla_{\phi X} Y - h(\phi X, Y) - \phi\nabla_{\phi X} \phi Y - \omega\nabla_{\phi X} \phi Y - Bh(\phi X, \phi Y) - Ch(\phi X, \phi Y)$$

$$+ \phi A_{\omega Y}\phi X + \omega A_{\omega Y}\phi X - B\nabla^{\perp}_{\phi X} \omega Y - C\nabla^{\perp}_{\phi X} \omega Y = 0 . \qquad (2.12)$$

By comparing to the tangent part and the normal part in (2.12), we get

$$\nabla_X \phi Y - A_{\omega Y} X - \phi\nabla_X Y - Bh(X, Y) - \nabla_{\phi X} Y - \phi\nabla_{\phi X} \phi Y - Bh(\phi X, \phi Y)$$

$$+ \phi A_{\omega Y}\phi X - B\nabla^{\perp}_{\phi X} \omega Y = 0 \qquad (2.13)$$

And

$$h(X, \phi Y) + \nabla^{\perp}_X \omega Y - \omega\nabla_X Y - Ch(X, Y) - h(\phi X, Y) - \omega\nabla_{\phi X} \phi Y - Ch(\phi X, \phi Y)$$

$$+ \omega A_{\omega Y}\phi X - C\nabla^{\perp}_{\phi X} \omega Y = 0 . \qquad (2.14)$$

By (2.13) and (1.18) we have (2.7). Also, by (2.14) and (1.19) we get (2.8). Q.E.D.

Lemma 2.4([1]). Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then we have

$$S(X, Y) = (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + \phi\{(\nabla_Y\phi)X - (\nabla_X\phi)Y\} - B\{(\nabla_X\omega)Y - (\nabla_Y\omega)X\} , \qquad (2.15)$$

for any $X, Y \in \Gamma(TM)$.

Lemma2.5. Let M be a CR-submanifold of a quasi Kaehlerian manifold $\overline{M}$. Then we have

$$S(X, Y) = A_{\omega Y}\phi X - \phi A_{\omega Y} X - A_{\omega X}\phi Y + \phi A_{\omega X} Y + ((\overline{\nabla}_{\phi X} J)Y - (\nabla_{\phi Y} J)X)^T$$

$$- \frac{1}{2}\phi(J[J, J](X, Y))^T - \frac{1}{2}B(J[J, J](X, Y))^{\perp} , \qquad (2.16)$$

for any $X, Y \in \Gamma(TM)$.

Proof: For any $X, Y \in \Gamma(TM)$. Taking into account (1.3), (1.11), (1.8), (1.9) and (1.17), we have

$$(\overline{\nabla}_X J)Y = \overline{\nabla}_X (\phi Y + \omega Y) - J(\nabla_X Y + h(X, Y))$$

$$= \nabla_X \phi Y + h(X, \phi Y) - A_{\omega Y} X + \nabla^{\perp}_X \omega Y$$

$$- \phi\nabla_X Y - \omega\nabla_X Y - Bh(X, Y) - Ch(X, Y) . \qquad (2.17)$$

By comparing to the tangent part and the normal part in (2.17), we obtain

$$((\overline{\nabla}_X J)Y)^T = \nabla_X \phi Y - A_{\omega Y} X - \phi\nabla_X Y - Bh(X, Y) \qquad (2.18)$$

and

$$((\overline{\nabla}_X J)Y)^{\perp} = h(X,\ \phi Y) + \nabla^{\perp}_X \omega Y - \omega\nabla_X Y - Ch(X,\ Y). \qquad (2.19)$$

Combining (1.18) and (2.18), we have

$$(\nabla_X \phi)Y = A_{\omega Y} X + Bh(X,\ Y) + ((\overline{\nabla}_X J)Y)^T. \qquad (2.20)$$

Combining (1.19) and (2.19), we get

$$(\nabla_X \omega)Y = -h(X,\ \phi Y) + Ch(X,\ Y) + ((\overline{\nabla}_X J)Y)^{\perp}. \qquad (2.21)$$

Taking account of (2.20) and (2.21), (2.15) becomes

$$S(X,\ Y) = A_{\omega Y}\phi X + ((\overline{\nabla}_{\phi X} J)Y)^T - A_{\omega X}\phi Y - ((\overline{\nabla}_{\phi Y} J)X)^T + \phi A_{\omega X} Y + \phi((\overline{\nabla}_Y J)X)^T$$
$$-\phi A_{\omega Y} X - \phi((\overline{\nabla}_X J)Y)^T - B((\overline{\nabla}_X J)Y)^{\perp} + B((\overline{\nabla}_Y J)X)^{\perp}. \qquad (2.22)$$

Combining (2.22) and (2.1), we obtain our conclusion (2.16).

Theorem2.1. Let M be a CR-submanifold of a quasi Kaehlerian manifold $\overline{M}$. Then M is normal if and only if we have

$$0 = A_{\omega Y}\phi X - \phi A_{\omega Y} X - A_{\omega X}\phi Y + \phi A_{\omega X} Y + ((\overline{\nabla}_{\phi X} J)Y - (\nabla_{\phi Y} J)X)^T$$
$$-\frac{1}{2}\phi(J[J,\ J](X,\ Y))^T - \frac{1}{2}B(J[J,\ J](X,\ Y))^{\perp}, \qquad (2.23)$$

for any $X,\ Y \in \Gamma(TM)$.

Proof: Taking account of Definition 1.4 and Lemma2.5, our conclusion holds. Q.E.D. Corollary2.1. Let M be a CR-submanifold of a Kaehlerian manifold $\overline{M}$. Then M is normal if and only if we have

$$A_{\omega Y}\phi X - \phi A_{\omega Y} X - A_{\omega X}\phi Y + \phi A_{\omega X} Y = 0, \qquad (2.24)$$

for any $X,\ Y \in \Gamma(TM)$. Proof: Since a Kaehlerian manifold $\overline{M}$ satisfies

$\overline{\nabla}_X J = 0,\ [J,\ J](X,\ Y) = 0$,

for any $X,\ Y \in \Gamma(TM)$, taking account of Theorem2.1, Corollary2.1 holds. Q.E.D.

Corollary2.2(Bejiancu[1]). Let M be a CR-submanifold of a Kaehlerian manifold $\overline{M}$. Then M is normal if and only if we have $A_{\omega Y}\phi X = \phi A_{\omega Y} X$, (2.25)

for any $X \in \Gamma(D),\ Y \in \Gamma(D^{\perp})$. Theorem2.2. Let M be a CR-submanifold of a quasi Kaehlerian manifold $\overline{M}$ and $[J,\ J](X,\ Y) \in \Gamma(\nu)$, (2.26)

for any $X,\ Y \in \Gamma(TM)$. Then M is normal if and only if we have

$$A_{\omega Y} X + \nabla_{\phi X} Y \in \Gamma(D^{\perp}) \qquad (2.27)$$

and

$$h(X,\ Y) \in \Gamma(\nu), \qquad (2.28)$$

for any $X \in \Gamma(D), \ Y \in \Gamma(D^{\perp})$.

Proof: For any $X \in \Gamma(D), \ Y \in \Gamma(D^{\perp})$. By using (2.26) in (2.16) we obatin

$$S(X, \ Y) = A_{\omega Y}\phi X - \phi A_{\omega Y} X + ((\overline{\nabla}_{\phi X} J)Y)^{T}. \qquad (2.29)$$

Taking into about (1.8), (1.9), (1.11) and (1.17), (1.3) becomes

$$(\overline{\nabla}_{\phi X} J)Y = -A_{\omega Y}\phi X + \nabla^{\perp}_{\phi X}\omega Y - \phi\nabla_{\phi X} Y - \omega\nabla_{\phi X} Y - Bh(\phi X, \ Y) - Ch(\phi X, \ Y). \qquad (2.30)$$

By comparing to the tangent part and the normal part in (2.30), we get

$$((\overline{\nabla}_{\phi X} J)Y)^{T} = -A_{\omega Y}\phi X - \phi\nabla_{\phi X} Y - Bh(\phi X, \ Y). \qquad (2.31)$$

From (2.29) and (2.31), we obtain

$$S(X, \ Y) = -\phi A_{\omega Y} X - \phi\nabla_{\phi X} Y - Bh(\phi X, \ Y). \qquad (2.32)$$

Suppose M is normal CR-submanifold of $\overline{M}$. For any $X \in \Gamma(D), \ Y \in \Gamma(D^{\perp})$, then from (2.32) and Definition1.4 we have $\phi(A_{\omega Y} X + \nabla_{\phi X} Y) = 0$ (2.33)

And $Bh(\phi X, \ Y) = 0$. (2.34)

From (2.33) we obtain (2.27), correspondingly, from (2.34) we get (2.28). Conversely, if (2.27) and (2.28) are sastified. Now, we shall prove $S = 0$ by means of the decomposition $TM = D \oplus D^{\perp}$. First, for any $X \in \Gamma(D), \ Y \in \Gamma(D^{\perp})$, from (2.27) we obtain (2.33), correspondingly, from (2.28) we get (2.34). Taking account of (2.33) and (2.34), (2.32) becomes $S(X, \ Y) = 0, \ \forall X \in \Gamma(D), \ Y \in \Gamma(D^{\perp})$. Next, for any $X, \ Y \in \Gamma(D)$, by using (2.26), (2.16) changes into $S(X, \ Y) = ((\overline{\nabla}_{\phi X} J)Y)^{T} - ((\overline{\nabla}_{\phi Y} J)X)^{T}$

$$= ((\overline{\nabla}_{JX} J)Y - (\overline{\nabla}_{JY} J)X)^{T}. \qquad (2.35)$$

From (2.4) and (2.26), (2.35) becomes $S(X, \ Y) = 0, \ \forall X, \ Y \in \Gamma(D)$. Finally, for any $X, \ Y \in \Gamma(D^{\perp})$, in accordance with (2.26), (2.16) changes over $S(X, \ Y) = -\phi A_{\omega Y} X + \phi A_{\omega X} Y \in \Gamma(D)$. (2.36)

$\forall Z \in \Gamma(D)$, on the basis of (2.36), (1.11) and (1.10), we have

$$g(S(X, \ Y), \ Z) = g(-\phi A_{\omega Y} X, \ Z) + g(\phi A_{\omega X} Y, \ Z)$$
$$= g(h(X, \ \phi Z), \ \omega Y) - g(h(Y, \ \phi Z), \ \omega Y). \qquad (2.37)$$

Using (2.28) in (2.37), we get

$$g(S(X, \ Y), \ Z) = 0. \qquad (2.38)$$

That is, $S(X, \ Y) = 0, \ \forall X, \ Y \in \Gamma(D)$.

From the above three conclusions we know $S(X, Y)=0$, for any $X, Y \in \Gamma(TM)$. Thus, the CR-submanifold M is normal. Q.E.D. Theorem2.3. Let M be a CR-submanifold of a quasi Kaehlerian manifold $\overline{M}$ with following conditions satisfing $\nabla_X Y \in \Gamma(D)$ (2.39)

And $h(X, Y) \in \Gamma(\nu)$, (2.40)

for any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$.Then M is mixed normal if and only if we have

$S^*(Y, X)=0$, (2.41)

for any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$.

Proof: For any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$. According to (1.18), (2.15) becomes

$S(X, Y)=\phi(\phi[X, Y]-[\phi X, Y])-B(\nabla_X \omega)Y+B(\nabla_Y \omega)X$. (2.42)

Taking into account (1.19), (2.8) and $B \circ C=0$, (2.42) changes into

$S(X, Y)=\phi(\phi[X, Y]-[\phi X, Y])-Bh(\phi X, Y)+B\omega A_{\omega Y}\phi X-B\omega\nabla_Y X$. (2.43)

Taking account of (1.23), (2.40) and $B \circ \omega=-Q$, (2.43) changes over

$S(X, Y)=\phi S^*(Y, X)-QA_{\omega Y}\phi X+Q\nabla_Y X$. (2.44)

$\forall U \in \Gamma(D^{\perp})$, combining (1.12), (1.10) and (2.40), we have

$g(QA_{\omega Y}\phi X, U)=g(A_{\omega Y}\phi X, U)=g(h(\phi X, U), \omega Y)=0$. (2.45)

(2.45) leads to $QA_{\omega Y}\phi X=0, \forall X \in \Gamma(D), Y \in \Gamma(D^{\perp})$. (2.46)

Combining (2.44) and (2.46), we get $S(X, Y)=\phi S^*(Y, X)+Q\nabla_Y X, \forall X \in \Gamma(D), Y \in \Gamma(D^{\perp})$. (2.47)

Suppose M is mixed normal CR-submanifold of $\overline{M}$. For any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$, then from (2.47) it follows

$$\phi S^*(Y, X)=0 \quad (2.48)$$

and

$$Q\nabla_Y X=0. \quad (2.49)$$

Based on (2.48) we obtain

$$S^*(Y, X) \in \Gamma(D^{\perp}), \quad (2.50)$$

for any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$. On the other hand, taking into account (2.39) and (2.49), (1.23) becomes

$$S^*(Y, X)=\nabla_Y \phi X-\nabla_{\phi X} Y-\phi[Y, X] \in \Gamma(D), \quad (2.51)$$

for any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$. Taking account of (2.50) and (2.51), we get that (2.41) holds.

Conversely, if (2.41) is sastified. For any $X \in \Gamma(D), Y \in \Gamma(D^{\perp})$, combinning (1.15) and (1.12), (1.23) changes into

$$S^*(Y,\ -\phi X) = [Y,\ X] - \phi[Y,\ -\phi X] = P[Y,\ X] - \phi[Y,\ -\phi X] + Q[Y,\ X]. \quad (2.52)$$

By using (2.41) in (2.52), we have $Q[Y,\ X] = 0,\ \forall X \in \Gamma(D),\ Y \in \Gamma(D^{\perp})$. (2.53)

From (2.53) and (2.39), we obtain $Q\nabla_Y X = 0,\ \forall X \in \Gamma(D),\ Y \in \Gamma(D^{\perp})$. (2.54)

Combining (2.41) and (2.54), (2.47) becomes $S(X,\ Y) = 0,\ \forall X \in \Gamma(D),\ Y \in \Gamma(D^{\perp})$. (2.55)

Relying on Definition 1.5, M is mixed normal. Q.E.D.

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