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On the Oscillation of Solutions of Fractional Vector Partial Differential Equations with Deviating Arguments

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Abstract

In this article, we investigate the oscillation of solutions of a class of fractional vector partial differential equations with deviating arguments of the form

$$D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}U(x,t)\right]=a(t)\Delta U(x,t)+\sum_{i=1}^{m}a_i(t)\Delta U(x,\rho_i(t))$$
$$-\sum_{j=1}^{k}p_j(x,t)f_j\left(\int_0^t(t-s)^{-\alpha}\left\|U(x,\sigma_j(s))\right\|ds\right)U(x,\sigma_j(t))$$
$$+F(x,t),(x,t)\in G=\Omega\times R_+,$$

subject to the boundary condition

$$\frac{\partial U}{\partial\gamma}+\mu(x,t)U(x,t)=0,\quad (x,t)\in\partial\Omega\times R_+.$$

We will establish the sufficient conditions for H-oscillation of solutions of given system, where H is a unit vector in R^n, $n\geq 1$. We also provide an example to illustrate the main results.

Keywords: Fractional, partial, oscillation, vector differential equation, deviating arguments.

1 Introduction

Fractional differential equations are now recognized as an excellent source of knowledge in modelling dynamical processes in self similar and porous structures, electrical networks, probability and statistics, visco elasticity, electro chemistry of corrosion, electro dynamics of complex medium, polymer rheology, industrial robotics, economics, biotechnology etc. See the recent monograph [2, 11-14, 16, 23, 29] for theory and applications of fractional differential equations. Oscillatory solution plays an important role in the quantitative and qualitative theory of fractional differential equations. There are several papers dealing with oscillation of scalar fractional ordinary differential equations [3-5, 9, 24, 27-28]. However, only a few results have appeared regarding the oscillatory behavior of scalar fractional partial differential equations, see [1, 18-22, 26] and the references cited there in.

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In 1970, Domslak introduced the concept of H-oscillation to investigate the oscillation of solutions of vector differential equations, where H is a unit vector in R^n. We refer the articles [6-7] for vector ordinary differential equations and [8, 15, 17, 25] for vector partial differential equations. To the present time, there exists almost no literature on oscillation results for vector fractional ordinary differential equations and vector fractional partial differential equations, particularly for vector fractional nonlinear partial differential equations. Motivated by this, we initiate the fractional order vector partial differential equations for delay equations.

Formulation of the problems: The oscillatory theory of fractional differential equation was introduced by Grace et al [9]

$$D_a^q x + f_1(t,x) = v(t) + f_2(t,x)$$
$$\lim_{t\to a^+} J_a^{1-q} x(t) = b,$$

where D_a^q denotes the Riemann-Liouville differential operator of q, where $0 < q < 1$.

Chen [4] and Han et al [28] studied the oscillation of the fractional differential equation with Liouville right sided fractional derivative of order α of the following form

$$\left(r(t)\left(D_-^\alpha y\right)^\eta(t)\right)' - q(t) f\left(\int_t^\infty (s-t)^{-\alpha} y(s)ds\right) = 0, \quad t > 0,$$
$$\left(r(t)g(D_-^\alpha(y(t))\right)' - p(t) f\left(\int_t^\infty (s-t)^{-\alpha} y(s)ds\right) = 0, \quad t > 0.$$

Prakash et al. [18] and Sadhasivam and Kavitha [21] investigated the fractional partial differential equation with Riemann-Liouville left sided definition on the half axis R_+ of the form

$$\frac{\partial}{\partial t}\left[r(t)D_{+,t}^\alpha u(x,t)\right] + q(x,t) f\left(\int_0^t (t-v)^{-\alpha} u(x,v)dv\right) = a(t)\Delta u(x,t), (x,t)\in\Omega\times R_+ = G,$$

with the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0, \quad (x,t)\in\partial\Omega\times R_+.$$

$$\frac{\partial}{\partial t}\left[p(t)g(D_{+,t}^\alpha u(x,t))\right] + \sum_{j=1}^m q_j(x,t) f_j\left(\int_0^t (t-s)^{-\alpha} u(x,s)ds\right) = a(t)\Delta u(x,t) + F(x,t),\ (x,t)\in\Omega\times R_+ = G,$$

subject to the boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} + \mu(x,t)u(x,t) = 0, \quad (x,t)\in\partial\Omega\times R_+.$$

To the best of our knowledge, nothing is known regarding the H-oscillatory behavior for the following class of vector fractional partial differential equations with forced term of the form

$$D_{+,t}^\alpha\left[r(t)D_{+,t}^\alpha U(x,t)\right] = a(t)\Delta U(x,t) + \sum_{i=1}^m a_i(t)\Delta U(x,\rho_i(t))$$
$$-\sum_{j=1}^k p_j(x,t) f_j\left(\int_0^t (t-s)^{-\alpha}\left\|U(x,\sigma_j(s))\right\|ds\right)U(x,\sigma_j(t))$$
$$+F(x,t), \quad (x,t)\in G = \Omega\times R_+,$$

$R_+ = (0,\infty)$, where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega, \alpha\in(0,1)$ is a constant, $D_{+,t}^\alpha$ is the Riemann-Liouville fractional derivative of order α of u with respect to t, Δ is the Laplacian operator in the Euclidean n - space R^n (ie) $\Delta u(x,t) = \sum_{r=1}^n \frac{\partial^2 u(x,t)}{\partial x_r^2}$ and $\left\|U(x,\sigma_j(s))\right\|$ is the usual Euclidean norm in R^n.

Equation (1.1) is supplemented with the following boundary conditions

$$\frac{\partial U(x,t)}{\partial \gamma} + \mu(x,t)U(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+, \tag{1.2}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)$ is positive continuous function on $\partial\Omega \times R_+$ and

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+. \tag{1.3}$$

In what follows, we always assume without mentioning that

(A_1) $r(t) \in C^{\alpha}(R_+;R_+), a(t), a_i(t) \in C(R_+;R_+), i = 1,2,...m$;

(A_2) $\sigma_j, \rho_i \in C(R_+;R), \lim_{t\to\infty} \sigma_j(t) = \lim_{t\to\infty} \rho_i(t) = \infty, i = 1,2,..m, j = 1,2,...,k$;

(A_3) $p_j \in C(\overline{G};R)$ and $p_j(t) = min_{x\in\overline{\Omega}} p_j(x,t), j \in I_k = \{1,2,...,k\}$;

(A_4) $F \in C(\overline{G};R^n), f_H(x,t) \in C(\overline{G};R)$ and $\int_{\Omega} f_H(x,t)dx \le 0$;

(A_5) $f_j \in C(R_+;R)$ are convex and non decreasing in R with $uf_j(u) > 0$ for $u \ne 0$ and there exist positive constants α_j such that $\frac{f_j(u)}{u} \ge \alpha_j$ for all $u \ne 0, j \in I_k$.

The study of H-oscillatory behavior of fractional partial differential equation is initiated in this paper. Our approach is to reduce multi-dimensional problems for (1.1) to one dimensional oscillation problems for scalar functional fractional differential inequalities. The purpose of this paper is to establish some new H-oscillation criteria for equation (1.1) with (1.2) and equation (1.1) with (1.3) by using a generalized Riccati technique and integral averaging method. Our results are essentially new.

2 Preliminaries

In this section, we give the definitions of H-oscillation, fractional derivatives and integrals and some notations which are useful throughout this paper. There are serveral kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left sided definition on the half-axis R_+. The following notations will be used for the convenience.

$u_H(x,t) = \langle U(x,t), H\rangle, f_H(x,t) = \langle F(x,t), H\rangle$,

$$V_H(t) = \frac{1}{|\Omega|}\int_{\Omega} u_H(x,t)dx, \quad where |\Omega| = \int_{\Omega} dx. \tag{2.1}$$

Definition: 2.1 By a solution of (1.1),(1.2) and (1.3) we mean a non trivial function $U(x,t) \in C^{2\alpha}(\overline{G};R^n) \cap C^2(\overline{G}\times[\hat{t}_{-1},\infty);R^n) \cap C(\overline{G}\times[\tilde{t}_{-1},\infty);R^n)$ and satisfies (1.1) on $\overline{G}$ and the boundary conditions (1.2) and (1.3), where $\hat{t}_{-1} = min\left\{0, \min_{1\le i\le m}\left\{\inf_{t\ge 0} \rho_i(t)\right\}\right\}$, $\tilde{t}_{-1} = min\left\{0, \min_{1\le j\le m}\left\{\inf_{t\ge 0} \sigma_j(t)\right\}\right\}$.

Definition: 2.2 Let H be a fixed unit vector in R^n. A solution $U(x,t)$ of (1.1) is said to be H-oscillatory in G if the inner product $\langle U(x,t), H\rangle$ has a zero in $\Omega\times(t,\infty)$ for any $t > 0$. Otherwise it is H-nonoscillatory.

Definition: 2.3 The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$D^{\alpha}_{+,t}u(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-v)^{-\alpha}u(x,v)dv, \tag{2.2}$$

provided the right hand side is pointwise defined on R_+ where Γ is the gamma function.

Definition: 2.4 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y: R_+ \to R$ on the half-axis R_+ is given by

$$I_+^{\alpha} y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} y(v) dv \quad for \quad t > 0, \tag{2.3}$$

provided the right hand side is pointwise defined on R_+.

Definition: 2.5 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$D_+^{\alpha} y(t) := \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \left(I_+^{\lceil \alpha \rceil - \alpha} y \right)(t) \quad for \quad t > 0, \tag{2.4}$$

provided the right hand side is pointwise defined on R_+ where $\lceil \alpha \rceil$ is the ceiling function of α .

Lemma: 2.1 [11] Let y be solution of (1.1) and

$$K(t) := \int_0^t (t-s)^{-\alpha} y(s) ds \quad for \quad \alpha \in (0,1) \quad and \quad t > 0. \tag{2.5}$$

Then

$$K'(t) = \Gamma(1-\alpha) D_+^{\alpha} y(t) \quad for \quad \alpha \in (0,1) \quad and \quad t > 0. \tag{2.6}$$

Lemma: 2.2 [10] If X and Y are nonnegative, then

$$mXY^{m-1} - X^m \le (m-1) Y^m, \tag{2.7}$$

where m is a positive integer.

3 H-Oscillation of the problem $(1.1), (1.2)$

We begin with the following Lemma.

Lemma: 3.1 Assume that $(A_1)-(A_5)$ hold. Let H be a fixed unit vector in R^n and $U(x,t)$ be a solution of (1.1).

(i)If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar fractional partial inequality

$$D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} u_H(x,t)\right] - a(t)\Delta u_H(x,t) - \sum_{i=1}^{m} a_i(t) \Delta u_H(x, \rho_i(t))$$
$$+ \sum_{j=1}^{k} p_j(t) f_j\left(\int_0^t (t-s)^{-\alpha} u_H(x, \sigma_j(s)) ds\right) u_H(x, \sigma_j(t)) \le f_H(x,t). \tag{3.1}$$

(ii)If $u_H(x,t)$ is eventually negative, then $u_H(x,t)$ satisfies the scalar fractional partial inequality

$$D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} u_H(x,t)\right] - a(t)\Delta u_H(x,t) - \sum_{i=1}^{m} a_i(t) \Delta u_H(x, \rho_i(t))$$
$$+ \sum_{j=1}^{k} p_j(t) f_j\left(\int_0^t (t-s)^{-\alpha} u_H(x, \sigma_j(s)) ds\right) u_H(x, \sigma_j(t)) \ge f_H(x,t). \tag{3.2}$$

Proof. Let $u_H(x,t)$ be eventually positive. Taking the inner product of (1.1) and H, we get

$$D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} \langle U(x,t), H\rangle\right] = a(t)\Delta\langle U(x,t), H\rangle + \sum_{i=1}^{m} a_i(t)\Delta\langle U(x,\rho_i(t)), H\rangle$$
$$- \sum_{j=1}^{k} p_j(x,t) f_j\left(\int_0^t (t-s)^{-\alpha} \left\|U(x,\sigma_j(s))\right\| ds\right) \langle U(x,\sigma_j(t)), H\rangle + \langle F(x,t), H\rangle,$$

that is,

$$D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} u_H(x,t)\right] = a(t)\Delta u_H(x,t) + \sum_{i=1}^{m} a_i(t) \Delta u_H(x, \rho_i(t))$$
$$- \sum_{j=1}^{k} p_j(x,t) f_j\left(\int_0^t (t-s)^{-\alpha} \left\|U(x,\sigma_j(s))\right\| ds\right) u_H(x,\sigma_j(t)) + f_H(x,t). \tag{3.3}$$

By (A_3), we have

$$p_j(x,t)f_j\left(\int_0^t (t-s)^{-\alpha}\left\|U(x,\sigma_j(s))\right\|ds\right)u_H(x,\sigma_j(t))$$

$$\geq p_j(t)f_j\left(\int_0^t (t-s)^{-\alpha}\left\|U(x,\sigma_j(s))\right\|ds\right)u_H(x,\sigma_j(t)),$$

since $f_j \in C(R_+,R), j=1,2..k,$ we have $u_H(x,\sigma_j(s)) \leq \left\|U(x,\sigma_j(s))\right\|$, therefore

$$p_j(t)f_j\left(\int_0^t (t-s)^{-\alpha}\left\|U(x,\sigma_j(s))\right\|ds\right)u_H(x,\sigma_j(t))$$

$$\geq p_j(t)f_j\left(\int_0^t (t-s)^{-\alpha}u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t)),\ j=1,2,...,k. \quad (3.4)$$

Using (3.4) in (3.3), we get

$$D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}u_H(x,t)\right]-a(t)\Delta u_H(x,t)-\sum_{i=1}^{m}a_i(t)\Delta u_H(x,\rho_i(t))$$

$$+\sum_{j=1}^{k}p_j(t)f_j\left(\int_0^t (t-s)^{-\alpha}u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t)) \leq f_H(x,t).$$

Similarly, let $u_H(x,t)$ be eventually negative, we easily obtain (3.2). The proof is complete.
The inner products of $(1.2),(1.3)$ with H yield the following boundary conditions.

$$\frac{\partial u_H(x,t)}{\partial\gamma}+\mu(x,t)u_H(x,t)=0, \quad (x,t)\in\partial\Omega\times R_+, \quad (1.2)'$$

$$u_H(x,t)=0, \quad (x,t)\in\partial\Omega\times R_+. \quad (1.3)'$$

Lemma: 3.2 Assume that $(A_1)-(A_5)$ hold. Let H be a fixed unit vector in R^n. If the scalar fractional partial inequality (3.1) has no eventually positive solutions and the scalar fractional partial inequality (3.2) has no eventually negative solutions satisfying the boundary conditions $(1.2)'$ or $(1.3)'$, then every solution $U(x,t)$ of the problem $(1.1),(1.2)$ or $(1.1),(1.3)$ is H-oscillatory in G. Proof. Suppose to the contrary that there is a H-nonoscillatory solution $U(x,t)$ of $(1.1),(1.2)$ or $(1.1),(1.3)$ in G, then $u_H(x,t)$ is eventually positive or $u_H(x,t)$ is eventually negative. If $u_H(x,t)$ is eventually positive, then by Lemma 3.1 $u_H(x,t)$ satisfies the boundry condition $(1.2)'$ or $(1.3)'$. This contradicts the hypothesis. The similar proof follows when $u_H(x,t)$ is eventually negative.

Theorem: 3.1 Assume that $(A_1)-(A_5)$ and

$(A_6)\ min_{j\in I_K}\{\sigma_j(t)\}=\sigma(t)\geq t$.

$(A_7)\ u_H(x,t)\geq L$ hold .

If the fractional differential inequality

$$D_+^{\alpha}\left[r(t)D_+^{\alpha}V_H(t)\right]+L\sum_{j=1}^{k}p_j(t)f_j(K_H(t))\leq 0, \quad (3.5)$$

has no eventually positive solutions and the fractional differential inequality

$$D_+^{\alpha}\left[r(t)D_+^{\alpha}V_H(t)\right]+L\sum_{j=1}^{k}p_j(t)f_j(K_H(t))\geq 0, \quad (3.6)$$

has no eventually negative solutions, then every solution $U(x,t)$ of (1.1) and (1.2) is H-oscillatory in G.

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1.1), (1.2) which is not a H-oscillatory in G. Without loss of genearality, we may assume that $u_H(x,t)>0$ in $\Omega\times[t_0,\infty)$ for some $t_0>0.$ Integrating (3.1) with respect to x over Ω, we obtain

$$\int_\Omega D_+^\alpha\left[r(t)D_+^\alpha u_H(t)\right]dx - a(t)\int_\Omega \Delta u_H(x,t)dx - \sum_{i=1}^m a_i(t)\int_\Omega \Delta u_H(x,\rho_i(t))dx$$

$$+\sum_{j=1}^k p_j(t)\int_\Omega f_j\left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t))dx \le \int_\Omega f_H(x,t)dx, \quad t\ge t_0. \tag{3.7}$$

Using Green's formula and boundary condition $(1.2)'$ yield that

$$\int_\Omega \Delta u_H(x,t)dx = \int_{\partial\Omega}\frac{\partial u_H(x,t)}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,t)u_H(x,t)dS \le 0, \quad t\ge t_0 \tag{3.8}$$

and

$$\int_\Omega \Delta u_H(x,\rho_i(t))dx = \int_{\partial\Omega}\frac{\partial u_H(x,\rho_i(t))}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,t)u_H(x,\rho_i(t))dS \le 0,$$

$$i=1,2,..m, t\ge t_0. \tag{3.9}$$

By using Jensen's inequality $(A_6),(A_7)$ and (2.1), we get

$$\int_\Omega f_j\left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))ds\right)u_H(x,\sigma_j(t))dx$$

$$\ge Lf_j\left(\int_\Omega\left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))ds\right)dx\right)$$

$$\ge Lf_j\left(\int_0^t (t-s)^{-\alpha}\left(\int_\Omega u_H(x,\sigma_j(s))dx\right)ds\right)$$

$$\ge L\int_\Omega dx f_j\left(\int_0^t (t-s)^{-\alpha}\left(\int_\Omega u_H(x,\sigma_j(s))dx(\int_\Omega dx)^{-1}\right)ds\right)$$

$$\ge L\int_\Omega dx f_j\left(\int_0^t (t-s)^{-\alpha} V_H(\sigma_j(s))ds\right)$$

$$\ge L\int_\Omega dx f_j(K_H(t)) \quad t\ge t_0. \tag{3.10}$$

Also by (A_4),

$$\int_\Omega f_H(x,t)dx \le 0. \tag{3.11}$$

In view of (2.1), (3.8)-(3.11), (3.7) yield

$$D_+^\alpha\left[r(t)D_+^\alpha V_H(t)\right] + L\sum_{j=1}^k p_j(t)f_j(K_H(t)) \le 0. \tag{3.12}$$

Therefore, $V_H(t)$ is an eventually positive solution of (3.5). This contradicts the hypothesis. The case where $u_H(x,t)<0$ in $\Omega\times[t_0,\infty)$ can be treated similarly and we are also getting a contradiction. The proof is now complete.

Theorem: 3.2 Suppose that the conditions $(A_1)-(A_7)$ and

$$\int_{t_0}^\infty\left(\frac{1}{r(s)}\right)ds = \infty \tag{3.13}$$

hold

Futhermore, assume that there exists a positive function $\delta\in C^\alpha((0,\infty);R_+)$ such that

$$\limsup_{\xi\to\infty}\int_{\xi_1}^{\xi}\left[L\tilde\delta(s)\sum_{j=1}^k \alpha_j\tilde p_j(s) - \frac{\tilde r(s)\left(\tilde\delta'(s)\right)^2}{4\Gamma(1-\alpha)\tilde\delta(s)}\right]ds = \infty, \tag{3.14}$$

where α_j are defined as in (A_5). Then every solution of $U(x,t)$ of the problem $(1.1),(1.2)$ is H-oscillatory in G.

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of the problem $(1.1),(1.2)$ which is not H-oscillatory in G. Without loss of generality we may assume that $u_H(x,t)>0$ in $\Omega\times[t_0,\infty)$ for some $t_0>0$.

That is, $V_H(t)$ is an eventually positive solution of (3.5). Then there exists $t_1 \ge t_0$ such that $V_H(t) > 0$ and $K_H(t) > 0$ for $t \ge t_1$. Therefore, it follows from (3.5) that

$$D_+^\alpha\left[r(t)D_+^\alpha V_H(t)\right] \le -L\sum_{j=1}^{k} p_j(t) f_j(K_H(t)) < 0 \quad for \quad t\in[t_1,\infty). \tag{3.15}$$

Thus $D_+^\alpha V_H(t) \ge 0$ or $D_+^\alpha V_H(t) < 0, t \ge t_1$ for some $t_1 \ge t_0$. We now claim that

$$D_+^\alpha V_H(t) \ge 0, \quad for \quad t \ge t_1. \tag{3.16}$$

Suppose not, then $D_+^\alpha V_H(t) < 0$ and there exists $t_2 \in [t_1,\infty)$ such that $D_+^\alpha V_H(t_2) < 0$. Since $r(t)D_+^\alpha V_H(t)$ is strictly decreasing on $[t_1,\infty)$. It is clear that

$$r(t)D_+^\alpha V_H(t) < r(t_2)D_+^\alpha V(t_2) := -c,$$

where $c > 0$ is a constant for $t \in [t_2,\infty)$. Therefore from (2.6), we have

$$\frac{K_H'(t)}{\Gamma(1-\alpha)} = D_+^\alpha V_H(t) < \left(\frac{-c}{r(t)}\right) \quad for \quad t\in[t_2,\infty).$$

Then, we get

$$\left(\frac{1}{r(t)}\right) \le -\frac{K_H'(t)}{c\Gamma(1-\alpha)} \quad for \quad t\in[t_2,\infty).$$

Integrating the above inequality from t_2 to t, we have

$$\int_{t_2}^{t}\left(\frac{1}{r(s)}\right)ds \le -\frac{K_H(t)-K_H(t_2)}{c\Gamma(1-\alpha)}$$

$$< \frac{K_H(t_2)}{c\Gamma(1-\alpha)} \quad for \quad t\in[t_2,\infty).$$

Letting $t \to \infty$, we get

$$\int_{t_2}^{\infty}\left(\frac{1}{r(s)}\right)ds \le \frac{K_H(t_2)}{c\Gamma(1-\alpha)} < \infty.$$

This contradicts (3.13). Hence $D_+^\alpha V_H(t) \ge 0$ for $t \in [t_1,\infty)$ holds.
Define the function $W(t)$ by the generalized Riccati substitution

$$W(t) = \delta(t)\frac{r(t)D_+^\alpha V_H(t)}{K_H(t)} \quad for \quad t\in[t_1,\infty). \tag{3.17}$$

Then we have $W(t) > 0$ for $t \in [t_1,\infty)$. From (2.6),(2.7), (3.5)and (A_5) it follows that

$$\begin{aligned} D_+^\alpha W(t) &= \frac{\delta(t)}{K_H(t)}D_+^\alpha\left[r(t)D_+^\alpha V_H(t)\right] + D_+^\alpha\left[\frac{\delta(t)}{K_H(t)}\right]r(t)D_+^\alpha V_H(t) \\ &\le -\delta(t)L\sum_{j=1}^{k}p_j(t)\frac{f_j(K_H(t))}{K_H(t)} + \left[\frac{K_H(t)D_+^\alpha\delta(t) - \delta(t)D_+^\alpha K_H(t)}{K_H^2(t)}\right]r(t)D_+^\alpha V_H(t) \\ &\le -L\delta(t)\sum_{j=1}^{k}\alpha_j p_j(t) + \frac{D_+^\alpha\delta(t)}{\delta(t)}W(t) - \frac{D_+^\alpha K_H(t)}{K_H(t)}W(t). \end{aligned} \tag{3.18}$$

Let $W(t) = \tilde{W}(\xi), \delta(t) = \tilde{\delta}(\xi), p_j(t) = \tilde{p}_j(\xi), K_H(t) = \tilde{K}_H(\xi)$.

Then $D_+^\alpha W(t) = \tilde{W}'(\xi), D_+^\alpha\delta(t) = \tilde{\delta}'(\xi)$. Then the above inequality becomes

$$\tilde{W}'(\xi) \le -L\tilde{\delta}(\xi)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(\xi) + \frac{\tilde{\delta}'(\xi)}{\tilde{\delta}(\xi)}\tilde{W}(\xi) - \frac{\tilde{K}_H'(\xi)}{\tilde{K}_H(\xi)}\tilde{W}(\xi)$$

$$\leq -L\tilde{\delta}(\xi)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(\xi)+\frac{\tilde{\delta}'(\xi)}{\tilde{\delta}(\xi)}\tilde{W}(\xi)-\frac{\Gamma(1-\alpha)\tilde{W}^2(\xi)}{\tilde{\delta}(\xi)\tilde{r}(\xi)}. \tag{3.19}$$

Taking $$m=2,\ X=\sqrt{\frac{\Gamma(1-\alpha)}{\tilde{\delta}(\xi)\tilde{r}(\xi)}}\tilde{W}(\xi),\ Y=\frac{1}{2}\sqrt{\frac{\tilde{r}(\xi)}{\Gamma(1-\alpha)\tilde{\delta}(\xi)}}\tilde{\delta}'(\xi). \tag{3.20}$$

Using Lemma 2.2 and (3.20) in (3.19), we have

$$\tilde{W}'(\xi)\leq -L\tilde{\delta}(\xi)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(\xi)+\frac{1}{4}\frac{\tilde{r}(\xi)\left(\tilde{\delta}'(\xi)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(\xi)}. \tag{3.21}$$

Integrating both sides of the above inequality from ξ_1 to ξ, we obtain

$$\int_{\xi_1}^{\xi}\left[L\tilde{\delta}(s)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(s)-\frac{1}{4}\frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(s)}\right]ds\leq \tilde{W}(\xi_1)-\tilde{W}(\xi)<\tilde{W}(\xi_1).$$

Taking the limit supremum of both sides of the above inequality as $\xi\to\infty$, we get

$$\limsup_{\xi\to\infty}\int_{\xi_1}^{\xi}\left[L\tilde{\delta}(s)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(s)-\frac{1}{4}\frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(s)}\right]ds<\tilde{W}(\xi_1)<\infty,$$

which contradicts (3.14) and completes the proof.

Theorem: 3.3 Suppose that the conditions $(A_1)-(A_7)$ and (3.13) hold. Futhermore, suppose that there exists a positive function $\delta\in C^{\alpha}((0,\infty);R_+)$ and a function $P\in C(D,R)$ where $D:=\{(t,s):t\geq s\geq t_0\}$ such that

1. $P(t,t)=0$ for $t\geq t_0$,
2. $P(t,s)>0$ for $(t,s)\in D_0$, where $D_0:=\{(t,s):t>s\geq t_0\}$ and P has a continuous and non-positive partial derivative $P_s'(t,s)=\dfrac{\partial P(t,s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{\xi\to\infty}\frac{1}{P(\xi,\xi_1)}\int_{\xi_1}^{\xi}P(\xi,s)\left[L\tilde{\delta}(s)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(s)-\frac{1}{4}\frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(s)}\right]ds=\infty, \tag{3.22}$$

where α_j are defined as in Theorem 3.2. Then all the solutions of $U(x,t)$ of the problem $(1.1),(1.2)$ is H-oscillatory in G. Proof. Suppose that $U(x,t)$ is H-nonoscillatory solution of $(1.1),(1.2)$. Without loss of generality we may assume that $u_H(x,t)$ is an eventually positive solution . Then $V_H(t)$ is an eventually positive solution of (3.5). Then proceeding as in the proof of Theorem 3.2, to get (3.21)

$$\tilde{W}'(\xi)\leq -L\tilde{\delta}(\xi)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(\xi)+\frac{1}{4}\frac{\tilde{r}(\xi)\left(\tilde{\delta}'(\xi)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(\xi)},$$

multiplying the previous inequality by $P(\xi,s)$ and integrating from ξ_1 to ξ for $\xi\in[\xi_1,\infty)$,

we obtain $$\int_{\xi_1}^{\xi}P(\xi,s)\left[L\tilde{\delta}(s)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(s)-\frac{1}{4}\frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(s)}\right]ds\leq-\left[P(\xi,s)\tilde{W}(s)\right]_{\xi_1}^{\xi}+\int_{\xi_1}^{\xi}P_s'(\xi,s)\tilde{W}(s)ds$$

$$\leq P(\xi,\xi_1)\tilde{W}(\xi_1)+\int_{\xi_1}^{\xi}P_s'(\xi,s)\tilde{W}(s)ds<P(\xi,\xi_1)\tilde{W}(\xi_1).$$

Therefore $$\frac{1}{P(\xi,\xi_1)}\int_{\xi_1}^{\xi}P(\xi,s)\left[L\tilde{\delta}(s)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(s)-\frac{1}{4}\frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(s)}\right]ds<\tilde{W}(\xi_1)<\infty,$$

which is a contradiction to (3.22).The proof is complete.

Corollary 3.1 Assume that the conditions of Theorem 3.3 hold with (3.22) replaced by

$$\limsup_{\xi\to\infty}\frac{1}{P(\xi,\xi_1)}\int_{\xi_1}^{\xi}P(\xi,s)L\tilde{\delta}(s)\sum_{j=1}^{k}\alpha_j\tilde{p}_j(s)ds=\infty,$$

$$\limsup_{\xi\to\infty}\frac{1}{P(\xi,\xi_1)}\int_{\xi_1}^{\xi}P(\xi,s)\frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{\Gamma(1-\alpha)\tilde{\delta}(s)}ds<\infty,$$

then every solution $U(x,t)$ of $(1.1),(1.2)$ is H-oscillatory in G. Next, we consider the case

$$\int_{t_0}^{\infty}\frac{1}{r(s)}ds<\infty, \tag{3.23}$$

which yields that (3.13) does not hold. In this case, we have the following result.

Theorem: 3.4 Suppose that the conditions $(A_1)-(A_7)$ and (3.23) hold and that there exists a positive function $\delta\in C^{\alpha}((0,\infty);R_+)$ such that (3.14) holds. Futhermore, assume that for every constant $\xi_T\geq\xi_0$, where $\xi_T=max\{\xi_3,\xi_4\}$

$$\int_{\xi_T}^{\infty}\left[\frac{1}{\tilde{r}(u)}\sum_{j=1}^{k}\alpha_j\int_{\xi_T}^{\xi}\tilde{p}_j(s)ds\right]du=\infty. \tag{3.24}$$

Then every solution of $\tilde{V}_H(\xi)$ of (3.5) is H-oscillatory or satisfies $\lim_{\xi\to\infty}\int_0^{\xi}(\xi-s)^{-\alpha}\tilde{V}_H(s)ds=0.$ Proof. Suppose that $U(x,t)$ is H-nonoscillatory solution of $(1.1),(1.2)$. Without loss of generality we may assume that $u_H(x,t)$ is an eventually positive solution . Then $V_H(t)$ is an eventually positive solution of (3.5). Then proceeding as in the proof Theorem 3.2, there are two cases for the sign of $D_+^{\alpha}V_H(t)$. The proof when $D_+^{\alpha}V_H(t)$ is eventually positive is similar to that of Theorem 3.2 and hence is omitted. Next, assume that $D_+^{\alpha}V_H(t)$ is eventually negative. Then there exists $t_3\geq t_2$ such that $D_+^{\alpha}V_H(t)<0$ for $t\geq t_3$. From (2.6), we get

$$K_H'(t)=\Gamma(1-\alpha)D_+^{\alpha}V_H(t)<0,\quad for\quad t\geq t_3.$$

Then $\tilde{K}_H'(\xi)=\Gamma(1-\alpha)\tilde{V}_H'(\xi)<0\quad for\quad \xi\geq\xi_3.$ Thus we get $\lim_{\xi\to\infty}\tilde{K}_H(\xi):=M_1\geq 0$ and $\tilde{K}_H(\xi)\geq M_1$. We claim that $M_1=0$. Assume not, that is, $M_1>0$ then from (A_5), we get

$$D_+^{\alpha}\left[r(t)D_+^{\alpha}V_H(t)\right]\leq -L\sum_{j=1}^{k}p_j(t)f_j\left(K_H(t)\right)$$

$$\leq -LM_1\sum_{j=1}^{k}\alpha_jp_j(t),\ for\quad t\in[t_3,\infty).$$

Let $r(t)=\tilde{r}(\xi),V_H(t)=\tilde{V}_H(\xi),\,p_j(t)=\tilde{p}_j(\xi)$.

Then $D_+^{\alpha}V_H(t)=\tilde{V}_H'(\xi),D_+^{\alpha}\left[r(t)D_+^{\alpha}V_H(t)\right]=\left(\tilde{r}(\xi)\tilde{V}_H'(\xi)\right)'$.

Using these values, the above inequality becomes

$\left(\tilde{r}(\xi)\tilde{V}_H'(\xi)\right)'\leq -LM_1\sum_{j=1}^{k}\alpha_j\tilde{p}_j(\xi),\ for\quad \xi\in[\xi_3,\infty).$ Integrating both sides of the last inequality from ξ_3 to ξ, we have

$$\int_{\xi_3}^{\xi}\left(\tilde{r}(s)\tilde{V}_H'(s)\right)'ds\leq -LM_1\sum_{j=1}^{k}\alpha_j\int_{\xi_3}^{\xi}\tilde{p}_j(s)ds$$

$$\tilde{r}(\xi)\tilde{V}_H'(\xi)\leq r(\xi_3)\tilde{V}_H'(\xi_3)-LM_1\sum_{j=1}^{k}\alpha_j\int_{\xi_3}^{\xi}\tilde{p}_j(s)ds\leq -k_1-LM_1\sum_{j=1}^{k}\alpha_j\int_{\xi_3}^{\xi}\tilde{p}_j(s)ds$$

$\leq -LM_1 \sum_{j=1}^{k} \alpha_j \int_{\xi_3}^{\xi} \tilde{p}_j(s)ds$. Hence from (2.6), we get $\frac{\tilde{K}'_H(\xi)}{\Gamma(1-\alpha)} = \tilde{V}'_H(\xi) \leq \frac{-LM_1 \sum_{j=1}^{k} \alpha_j \int_{\xi_3}^{\xi} \tilde{p}_j(s)ds}{\tilde{r}(\xi)}$.

Integrating the last inequality from ξ_4 to ξ, we get $\tilde{K}_H(\xi) \leq K_H(\xi_4) - \Gamma(1-\alpha) LM_1 \int_{\xi_4}^{\xi} \frac{\sum_{j=1}^{k} \alpha_j \int_{\xi_3}^{\xi} \tilde{p}_j(s)ds}{\tilde{r}(u)} du.$

Letting $\xi \to \infty$, from (3.24), we get $\lim_{\xi\to\infty} \tilde{K}_H(\xi) = -\infty$. This contradicts $\tilde{K}_H(\xi) > 0$. Therefore we have $M_1 = 0$, that is, $\lim_{\xi\to\infty} \tilde{K}_H(\xi) = 0$. That is, $\lim_{\xi\to\infty} \int_0^{\xi} (\xi - s)^{-\alpha} \tilde{V}_H(s)ds = 0$. Hence the proof.

4 H-Oscillation of the problem (1.1),(1.3)

In this section we establish sufficient conditions for the oscillation of all solutions of (1.1),(1.3). For this we need the following:The smallest eigen value β_0 of the Dirichlet problem. $\Delta\omega(x) + \beta\omega(x) = 0 \quad in \quad \Omega$, $\omega(x) = 0 \quad on \quad \partial\Omega$, is positive and the corresponding eigen function $\phi(x)$ is positive in Ω.

Theorem: 4.1 Let all the conditions of Theorem 3.2 and 3.3 be hold. Then every solution of $U(x,t)$ of (1.1) and (1.3) H-oscillates in G. Proof. Suppose that $U(x,t)$ is a H-nonoscillatory solution of (1.1) and (1.3). Without loss of generality we may assume that $u_H(x,t) > 0$, in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Multiplying both sides of the Equation (3.1) by $\phi(x) > 0$ and then integrating with respect to x over Ω.,

we obtain for $t \geq t_1$, $\int_\Omega D_+^\alpha \left[r(t) D_+^\alpha u_H(x,t) \right] \phi(x)dx - a(t)\int_\Omega \Delta u_H(x,t)\phi(x)dx - \sum_{i=1}^{m} a_i(t) \int_\Omega \Delta u_H(x,\rho_i(t))\phi(x)dx$

$$+\sum_{j=1}^{k} p_j(t) \int_\Omega f_j \left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))ds \right) u_H(x,\sigma_j(t)\phi(x)dx \leq \int_\Omega f_H(x,t)\phi(x)dx. \tag{4.1}$$

Using Green's formula and boundary condition $(1.3)'$ it follows that

$$\int_\Omega \Delta u_H(x,t)\phi(x)dx = \int_\Omega u_H(x,t)\Delta\phi(x)dx = -\beta_0 \int_\Omega u_H(x,t)\phi(x)dx \leq 0, \quad t \geq t_1 \tag{4.2}$$

and

$$\int_\Omega \Delta u_H(x,\rho_i(t))\phi(x)dx = \int_\Omega u_H(x,\rho_i(t))\Delta\phi(x)dx = -\beta_0 \int_\Omega u_H(x,\rho_i(t))\phi(x)dx \leq 0,$$
$$t \geq t_1, i = 1,2,...m. \tag{4.3}$$

By using and Jensen's inequality, (A_6) and (A_7) we get $\int_\Omega f_j \left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))ds \right) u_H(x,\sigma_j(t))\phi(x)dx$

$$\geq Lf_j \left(\int_\Omega \left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))\phi(x)ds \right) dx \right) \qquad \geq Lf_j \left(\int_0^t (t-s)^{-\alpha} \left(\int_\Omega u_H(x,\sigma_j(s))\phi(x)dx \right) ds \right)$$

$\geq L\int_\Omega \phi(x)dx f_j \left(\int_0^t (t-s)^{-\alpha} \left(\int_\Omega u_H(x,\sigma_j(s))\phi(x)dx (\int_\Omega \phi(x)dx)^{-1} \right) ds \right)$. Set

$$V_H(t) = \int_\Omega u_H(x,t)\phi(x)dx \left(\int_\Omega \phi(x)dx \right)^{-1}, \quad t \geq t_1. \tag{4.4}$$

Therefore, $\int_\Omega f_j \left(\int_0^t (t-s)^{-\alpha} u_H(x,\sigma_j(s))ds \right) u_H(x,\sigma_j(t))\phi(x)dx \geq L\int_\Omega \phi(x)dx f_j(K_H(t))$, $t \geq t_1, j \in I_m$. (4.5)

By (A_5), $\int_\Omega f_H(x,t)\phi(x)dx \leq 0.$ (4.6)

In view of (4.4), (4.2)-(4.6), (4.1) yields $D_{+}^{\alpha}\left[r(t)D_{+}^{\alpha}V_H(t)\right]+L\sum_{j=1}^{k}p_j(t)f_j(K_H(t))\le 0,$ (4.7)

for $t\ge t_1$. Rest of the proof is similar to that of Theorems 3.2 and 3.3, and hence the details are omitted.

Corollary 4.1 If the inequality (4.7) has no eventually positive solutions, then every solution $U(x,t)$ of (1.1) and (1.3) is H-oscillatory in G.

Corollary 4.2 Let the conditions of Corollary 3.1 hold; then every solution $U(x,t)$ of (1.1) and (1.3) is H-oscillatory in G.

Theorem: 4.2 Let the conditions of Theorem 3.4 hold; Then every solution $\tilde{V}_H(\xi)$ of (4.7) is H-oscillatory or satisfies $\lim_{\xi\to\infty}\int_0^{\xi}(\xi-s)^{-\alpha}\tilde{V}_H(s)ds=0$. The proofs of Corollaries 4.1 and 4.2 and Theorems 4.2 are similar to that of in Section 3 and hence the details are omitted.

5 Examples

In this section we give an example to illustrate the results established in Sections 3. **Example 1.** Consider the vector fractional partial differential equation

$$D_{+,t}^{\frac{1}{3}}\left[t^{\frac{2}{3}}D_{+,t}^{\frac{1}{3}}U(x,t)\right]=\frac{1}{4}t^{\frac{2}{3}}\Delta U(x,t)+\left(\frac{2\pi}{\Gamma(\frac{1}{3})^2}t^{\frac{1}{3}}+\frac{3}{4}t^{\frac{2}{3}}\right)\Delta U(x,t-\pi)$$

$$-\frac{1}{\sqrt{3}}\left(\int_0^t(t-s)^{\frac{-1}{3}}\left\|U\left(x,s-\frac{\pi}{2}\right)\right\|ds\right)U\left(x,t-\frac{\pi}{2}\right)+F(x,t), \quad (5.1)$$

$(x,t)\in G$, where $G=(0,\pi)\times(0,\pi)\times(0,\infty)$, with the boundary condition

$$U(0,t)=\begin{pmatrix}u_1(0,t)\\u_{2,}(0,t)\end{pmatrix}=U(\pi,t)=\begin{pmatrix}u_1(\pi,t)\\u_2(\pi,t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\quad t\ge 0. \quad (5.2)$$

Here $\alpha=\frac{1}{3}, m=1, k=1, n=2, r(t)=t^{\frac{2}{3}}, p_1(x,t)=\frac{1}{\sqrt{3}}$, $a(t)=\frac{1}{4}t^{\frac{2}{3}}$, $a_1(t)=\frac{2\pi}{\Gamma\left(\frac{1}{3}\right)^2}t^{\frac{1}{3}}+\frac{3}{4}t^{\frac{2}{3}}, \rho_1(t)=\pi, \sigma_1(t)=\frac{\pi}{2}$,

$$F(x,t)=\begin{pmatrix}\frac{2\pi}{\sqrt{3}(\Gamma(\frac{1}{3}))^2}t^{\frac{1}{3}}\sin x\cos t\\ \frac{(\Gamma(\frac{1}{3}))^2}{2\pi}+\frac{3\sqrt{3}}{2}t^{\frac{2}{3}}\end{pmatrix}$$

and $f_1(u)=u$. It is easy to see that $p_1(t)=min_{x\in\bar{\Omega}}p_1(x,t)=min_{x\in[0,\pi]}\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}$.

Let $H=e_1=\begin{pmatrix}1\\0\end{pmatrix}$, we observe that $f_{e_1}(x,t)=\frac{2\pi}{\sqrt{3}(\Gamma(\frac{1}{3}))^2}t^{\frac{1}{3}}\sin x\cos t$ and

$$\int_\Omega f_{e_1}(x,t)dx = \frac{4\pi}{\sqrt{3}(\Gamma(\frac{1}{3}))^2} t^{\frac{1}{3}} \cos t$$

$$\leq 0, \ \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}.$$

Take $\xi_1 = 1, \alpha_1 = 1, \tilde{\delta}(s) = s.$ It is clear that conditions $(A_1)-(A_7)$ and (3.13) hold. Therefore,

$$\int_{\xi_1}^{\xi} \left[L\tilde{\delta}(s)\alpha_1 \tilde{p}_1(s) - \frac{\tilde{r}(s)\left(\tilde{\delta}'(s)\right)^2}{4\Gamma(1-\alpha)\tilde{\delta}(s)} \right] ds = \int_1^{\xi} \left[L\frac{s}{\sqrt{3}} - \frac{1}{4\Gamma(\frac{2}{3})s^{\frac{1}{3}}} \right] ds \to \infty \quad as \quad \xi \to \infty.$$

Thus all the conditions of Theorem 3.2 are satisfied. Hence, it follows that every solution $U(x,t)$ of $(5.1),(5.2)$ is e_1-oscillatory in $G.$ Infact $U(x,t) = \begin{pmatrix} \sin x \sin t \\ \sqrt{3} \end{pmatrix},$ is one such solution of the problem (5.1) and (5.2). We note that the above solution $U(x,t)$ is not e_2 – oscillatory in G, where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

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