American Review of Mathematics and Statistics
December 2016, Vol. 4, No.2, pp. 82-91
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v4n1a9
URL: https://doi.org/10.15640/arms.v4n1a9

Optimal Reliability Allocation

Constantin Udrişte[1], Saad Abbas Abed[2], & Ali Sapeeh Rasheed[3]

Abstract

A system (mechanical, electrical, computer hardware and software etc) is generally designed as an assembly of subsystems, each with its own reliability attributes. The cost of the system is the sum of the costs for all the subsystems. This paper examines possible approaches to allocate the reliability values based on minimization of the total cost on the intersection between Tzitzeica semispace and a unit hypercube. The original results include: (i) a critical point is a fixed point of a suitable application, (ii) theorems for restoring Riemannian convex functions; (iii) the first cost with exponential behavior is Euclidean convex; the second cost with exponential behavior is Riemannian convex; our particular posynomial cost is Euclidean convex, (iv) an additively decomposable cost function is convex on a product Riemannian manifold.

Keywords: Optimization, reliability, allocation, Tzitzeica hypersurfaces, Riemannian convexity.

2010 MSC: 49K45, 62N05, 60K30

1. Introduction

We look for optimal reliability allocation problem as a mathematical problem though its origins are in engineering. The assignation of reliability values between the various subsystems and elements can be made on the basis of complexity, criticality, estimated achievable reliability, or any other factors considered appropriate by the analyst making the allocation. Many systems are implemented by using a set of interconnected subsystems. While the architecture of the overall system can often be fixed, individual subsystems may be implemented differently. A designer needs to either achieve the target reliability while minimizing the total cost, or maximize the reliability while using only the available budget. Intuitively, some of the lowest reliability components may need special attention to raise the overall reliability level.

Such an optimization problem may arise while designing a complex software or a computer system. Such problems also arise in mechanical or electrical systems. A number of studies have examined such problems [1], [4]. The model allocates reliability to a component according to the cost of increasing its reliability. The most costly components (with cost representing volume, cost, weight or any other quantity of concern) will be assigned the lowest increases in reliability. With this approach reliability can now be allocated to the components of any type of system, complex or not, and for a mixture of failure distributions for the components of the system.Two major factors have contributed to this situation. First, the model requires the system's analytical reliability equation as an input. Although this poses no major problem in simple systems, it can become quite a challenge (and very time consuming) in complex systems.

[1] Department of Mathematics and Informatics, Faculty of Applied Sciences, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania.
[2] Department of Computer Sciences, Faculty of Education, Iraqia University Baghdad, Iraq.
[3] Department of Mathematics and Informatics, Faculty of Applied Sciences, University of Wasit, Faculty of Science, Iraq.

Second, the model also requires cost as a function of the component's reliability as an input, and this is not always available to engineers. The parameters of the proposed cost function can be altered, allowing the engineers to investigate different allocation scenarios. Thereafter, reliability and design engineers can decide and plan on how to achieve the assigned minimum required reliabilities for each of the components.

2. Optimization for series system

Consider a series system consisting of n components connected reliability-wise in series. We use the notations: $0 \le R_i \le 1$ is the reliability of component i; $C_i(R_i)$ is the cost of component i; $C(R_1,...,R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ is the total system cost, where $a_i > 0$; R_s is the system reliability; R_G is the system reliability goal.

In general the functionality of each subsystems can be unique, however there can be several choices for, many of the subsystems providing the same functionality, but differently reliability levels. The objective is to allocate reliability to all or some of the components of that system, in order to meet that goal with a minimum cost. An important problem P is formulated as a nonlinear programming problem, *with additively decomposable cost function and a nonlinear constraint*:

P: Find

$$\min\ C(R_1,...,R_n) = \sum_{i=1}^{n} a_i C_i(R_i),\ a_i > 0,$$

subject to

$$R_s = \prod_{i=1}^{n} R_i \ge R_G,\ 0 < R_i \le 1,\ i = 1,2,...,n.$$

It is reasonable to assume that the partial cost function $C_i(R_i)$ satisfies some conditions [6]: differentiable, positive function, increasing $\left(\Rightarrow \frac{dC_i}{dR_i} \ge 0 \right)$.

The Euclidean convexity of the the partial cost function $C_i(R_i)$ is equivalent to the fact that its derivative $\frac{dC_i}{dR_i}$ is monotonically increasing, i.e., $\frac{d^2 C_i}{dR_i^2} \ge 0$.

The foregoing formulation is designed to achieve a minimum total system cost, subject to R_G, a lower limit on the system reliability.

2.1. Convexity of Tzitzeica hypersurfaces

We use the space R^n with coordinates $(R_1,...,R_n)$ and the constant $c > 0$. The constant level sets $\prod_{i=1}^{n} R_i = c$ attached to the foregoing constraint function

$$R_s(R_1,...,R_n) = \prod_{i=1}^{n} R_i$$

are Tzitzeica hypersurfaces in R^n. Since

$$\frac{\partial^2 R_s}{\partial R_i^2} = 0\,,\ \frac{\partial^2 R_s}{\partial R_i \partial R_j} = \frac{R_s}{R_i R_j}\,,$$

we find

$$d^2 R_s = 2R_s \sum_{i<j} \frac{dR_i}{R_i}\frac{dR_j}{R_j} = R_s\left(\left(\sum_{i=1}^{n}\frac{dR_i}{R_i}\right) - \sum_{i=1}^{n}\frac{dR_i^2}{R_i^2}\right).$$

The restriction of this Hessian to the tangent hyperplane $c\sum_{i=1}^{n}\frac{dR_i}{R_i} = 0$, i.e.,

$$d^2 R_s = -c\left(\sum_{i=1}^{n}\frac{dR_i^2}{R_i^2}\right)$$

is negative definite. Consequently, the second fundamental form is positive definite and hence the Tzitzeica hypersurface is convex.

2.2.Kuhn-Tucker necessary conditions

The associated Lagrange function is

$$L = \sum_{i=1}^{n} a_i C_i(R_i) + \lambda\left(R_G - \prod_{i=1}^{n} R_i\right) + \sum_{i=1}^{n}\alpha_i(1 - R_i).$$

The Kuhn-Tucker necessary conditions, for a point $(R_1,...,R_n)$ to be a minimum point, are:

$$\frac{\partial L}{\partial R_i} \geq 0\,,\ R_i > 0 \text{ and } R_i\frac{\partial L}{\partial R_i} = 0\,,\ i = 1,...,n\,,\ \Rightarrow \frac{\partial L}{\partial R_i} = 0\,;$$

$$\frac{\partial L}{\partial \lambda} = R_G - \prod_{j=i}^{n} R_i \leq 0\,,\ \lambda \geq 0 \text{ and } \lambda\left(R_G - \prod_{i=1}^{n} R_i\right) = 0 \Rightarrow R_G = \prod_{i=1}^{n} R_i\,;$$

$$\frac{\partial L}{\partial \alpha_i} = (1 - R_i) \geq 0 \text{ , } \alpha \ \geq 0 \text{ and } \alpha_i (1 - R_i) = 0 \Rightarrow \alpha_i = 0 \text{ , } R_i < 1 \text{ .}$$

The equations of critical points are $\frac{\partial L}{\partial R_i} = 0$. The constraints $R_i \leq 1$ are inactive, i.e., $0 < R_i < 1$. The constraint $\prod_{i=1}^{n} R_i \geq R_G$ is active, i.e., $\prod_{i=1}^{n} R_i = R_G$.

From the relations (critical points condition and active constraint)

$$a_i \frac{dC_i}{dR_i} - \lambda R_1 ... R_{i-1} R_{i+1} = 0 \text{ , } R_G = \prod_{i=1}^{n} R_i \text{ ,}$$

we find

$$a_i R_i \frac{dC_i}{dR_i} = \lambda \prod_{j=1}^{n} R_j = \lambda R_G \text{ ,}$$

and hence

$$R_i = \frac{\lambda R_G}{a_i \frac{dC_i}{dR_i}} \text{ , } \lambda^n = \frac{1}{R_G^{n-1} \prod_{j=1}^{n} \frac{dC_j}{dR_j}} \text{ .}$$

Proposition 2.1. *If* $(R_1, ..., R_n)$ *is a solution of the foregoing problem, then it is a fixed point of the application*

$$\left(\frac{\lambda R_G}{a_1 \frac{dC_1}{dR_1}}, ..., \frac{\lambda R_G}{a_n \frac{dC_n}{dR_n}} \right).$$

2.3. Sufficient conditions

We consider the Lagrange function

$$L = \sum_{i=1}^{n} a_i C_i (R_i) + \lambda \left(R_G - \prod_{i=1}^{n} R_i \right).$$

At critical point $(R_1, ..., R_n)$, we compute

$$d^2 L = \sum_{i=1}^{n} a_i d^2 C_i (R_i) - \lambda R_G \sum_{i=1}^{n} \frac{dR_i}{R_i} \text{ .}$$

Since $R_1 \ldots R_n = R_G$, we find the tangent hyperplane $\sum_{i=1}^{n} \frac{dR_i}{R_i} = 0$. In this way, at critical point, the Hessian is restricted to

$$d^2 L = \sum_{i=1}^{n} a_i d^2 C_i(R_i) = d^2 C(R_1, \ldots, R_n).$$

If $d^2 C > 0$, then the critical point $(R_1, \ldots, R_n)$ is a minimum point for the total cost.

3. Riemannian convexity

For details regarding the Riemannian convexity, see the papers [2], [7], [8].

Definition 3.1. *Let (M, g) be an n-dimensional Riemannian manifold and $f : M \to \mathbb{R}$ be a C^2 function. The function f is called Riemannian convex if one of the equivalent conditions is satisfied: (i) $f(\gamma_{xy}(t)) \le (1-t) f(x) + tf(y)$ for any geodesic $\gamma_{xy}(t)$, $t \in [0,1]$; (ii) the matrix function $Hess\, f = \left(\frac{\partial f}{\partial x^i \partial x^j} - \Gamma_{ij}^h \frac{\partial f}{\partial x^h} \right)$ is positive semidefinite, where Γ_{ij}^h is the connection induced by the Riemannian metric g_{ij}.*

3.1. Convex functions on $(\mathbb{R}, g(x))$

A Riemannian metric on $\mathbb{R}$ is any positive function $g(x)$. The Riemannian metric $g(x)$ on $\mathbb{R}$ determines a linear connection $\Gamma(x) = \frac{d}{dx} \ln g(x)$. If the Riemannian metric $g(x)$ is C^∞ then the connection $\Gamma(x)$ is C^∞.

For any C^2 function on $\mathbb{R}$, the Hessian means the function

$$Hess\, f(x) = f''(x) - \Gamma(x)\, f'(x).$$

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. If $f'(x) \ne 0$, then f is linear affine with respect to $g(x) = (cf'(x))^2$, where c is a constant.*

Hint $Hess\, f(x) = 0$.

Theorem 3.2. *A C^2 function $f : \mathbb{R} \to \mathbb{R}$ is convex on $(\mathbb{R}, g(x))$ if and only if the function $u(x) = \frac{f'(x)}{\sqrt{g(x)}}$ is increasing.*

Hint $Hess\, f(x) \ge 0$. For details, see the paper [8].

3.2.Restoring convex functions

Theorem 3.3. *If* $f_i : \mathrm{M} \to \mathrm{R}, \mathrm{i} = 1, ..., \mathrm{n},$ *are convex functions on the Riemannian manifold* $(M, g(x))$ *and* $c_i \geq 0,$ *then the function* $\sum_{\mathrm{i}=1}^{n} c_i f_i(x)$ *is convex on* $(M, g(x))$.

Theorem 3.4. *If* $f_i(x_i),\ i = 1,..,n,$ *are convex functions on the Riemannian manifolds* $(M_i, g(x_i))$*, respectively, and* $c_i \geq 0$*, then the function* $\sum_{i=1}^{n} c_i f_i(x_i)$ *is convex on the product manifold* $(M_1 \times ... \times M_n, \oplus_{i=1}^{n} g_i(x_i))$.

For other properties regarding the Riemannian convexity, see the papers [2], [7], [8].

4. Three significant reliability cost models

There is always a cost associated with changing a design, use of high quality materials, retooling costs, administrative fees, or other factors. The cost increases as the allocated reliability approaches the maximum achievable reliability. This is a reliability value that is approached asymptotically as the cost increases but is never actually reached. The cost increases as the allocated reliability departs from the minimum or current value of reliability. It is assumed that the reliabilities for the components will not take values any lower than they already have. Depending on the optimization, a component's reliability may not need to be increased from its current value but it will not drop any lower. The cost is a function of the range of improvement, which is the difference between the component's initial reliability and the corresponding maximum achievable reliability. This means that it is easier to increase the reliability of a component from a lower initial value. Before attempting at improving the reliability, the cost as a function of reliability for each component must be obtained. Otherwise, the design changes may result in a system that is needlessly expensive or overdesigned. Development of the cost of reliability relationship offers the engineer an understanding of which components or subsystems to improve. The first step is to obtain a relationship between the cost of improvement and reliability. The second step is to model the cost as a function of reliability. The preferred approach would be to formulate the cost function from actual cost data. This can be done taking the past data. However, there are many cases where no such information is available. For this reason, a general behavior model of the cost versus the component reliability can be developed for performing reliability optimization. The objective of cost functions is to model an overall cost behavior for all types of components. But, it is impossible to formulate a model that is precisely applicable to every situation. However, one of the reliability cost models available can be used depending on situation. All these models can be tried and one which is suitable to component or situation can be adopted.

4.1.Exponential behavior model

Let $0 < R_i < 1$, $i = 1,2,...,n$ and a_i, b_i be constants. The most important cost function has an exponential behavior. It was proposed by [5] (see also [3]) in the form

$$C_i(R_i) = a_i \exp\left(\frac{b_i}{1 - R_i}\right),\ i = 1,2,...,n\,.$$

Let $a_i > 0$, $b_i > 0$. After computation, we find

$$\frac{dC_i}{dR_i} = \frac{a_i b_i}{(R_i - 1)^2} \exp\left(\frac{-b_i}{R_i - 1}\right) > 0 .$$

$$Hess\ C_i = (b_i + 2 - 2R_i) \frac{a_i b_i}{(R_i - 1)^4} \exp\left(\frac{-b_i}{R_i - 1}\right) > 0 .$$

Consequently, each $C_i(R_i)$ is an increasing and convex function (in Euclidean sense). The total cost $C(R_1,\dots,R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ has similar properties.

4.2. Exponential behavior model with feasibility factor

Let $0 < f_i < 1$ be a feasibility factor, $R_{i,\min}$ be minimum reliability and $R_{i,\max}$ be maximum reliability. Another important cost function, with exponential behavior, is given by

$$C_i(R_i) = \exp\left((1 - f_i)\frac{R_i - R_{i,\min}}{R_{i.\max} - R_i}\right),\ R_{i,\min} \le R_i \le R_{i,\max},\ i = 1,2,\dots,n .$$

Since

$$\frac{dC_i}{dR_i} = \frac{(1 - f_i)(R_{i,\max} - R_{i,\min})}{(R_{i,\max} - R_i)^2}\ \exp\left((1 - f_i)\frac{R_i - R_{i,\min}}{R_{i,\max} - R_i}\right) > 0,$$

the function $C_i(R_i)$ is increasing. On the other hand

$$Hess\ C_i = \left(\frac{(1 - f_i)^2 (R_{i,\max} - R_{i,\min})^2}{(R_{i,\max} - R_i)^4} - \frac{2(1 - f_i)(R_{i,\max} - R_{i,\min})}{(R_{i,\max} - R_i)^3}\right)\exp\left((1 - f_i)\frac{R_i - R_{i,\min}}{R_{i,\max} - R_i}\right).$$

Consequently: (1) the graph of $C_i(R_i)$ has an inflection point at

$$R_i = \frac{(1 + f_i)R_{i,\max} + (1 - f_i)R_{i,\min}}{2};$$

(2) each $C_i(R_i)$ is a convex function (in Euclidean sense) for

$$R_i > \frac{(1 + f_i)R_{i,\max} + (1 - f_i)R_{i,\min}}{2};$$

(3) each $C_i(R_i)$ is a concave function (in Euclidean sense) for

$$R_i < \frac{(1+f_i)R_{i,\max} + (1-f_i)R_{i,\min}}{2}.$$

The total cost $C(R_1,...,R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ has similar properties.

Let us find a Riemannian metric $g_i(R_i)$ on $\mathbb{R}$ such that the function $C_i(R_i)$ to be convex on the Riemannian manifold $(\mathbb{R}, g_i(R_i))$. According the previous Section, and the Euclidean Hessian, it is enough to fix the connections $\Gamma_i(R_i) = \dfrac{2}{R_{i,\max} - R_i}$ and hence the metrics $g_i(R_i) = (R_{i,\max} - R_i)^{-4}$, for each index $i = 1, \dots, n.$ In this case, for each index, the ODE of geodesics

$$\ddot{R}_i(t) + \frac{2}{R_{i,\max} - R_i(t)} \dot{R}_i(t)\, \dot{R}_i(t) = 0$$

admits the general solution (Cartesian explicit form)

$$R(t) = R_{i\max} - \frac{1}{c_1 t + c_2};$$

the ODE of linear affine functions

$$f''(R_i) - \frac{2}{R_{i,\max} - R_i} f'(R_i) = 0$$

admits the general solution

$$f(R_i) = \frac{c_1}{R_{i\max} - R_i} + c_2.$$

Proposition 4.1. *The total cost* $C(R_1,...,R_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ *is convex on the product Riemannian manifold* $(\mathbb{R} \times ... \times \mathbb{R}, \oplus_{i=1}^{n} g_i(R_i))$.

4.3. Posynomial behavior model

We consider the posynomial cost (see also [7])

$$C(R) = c_1 R_1 + c_2 R_2 + c_3 R_1^{-a} R_2^{-b}, \quad 0 < R_i \le 1, \; c_i > 0, a > 0, b > 0.$$

To find $\min\ C(R)$, we look first for critical points

$$\frac{\partial C}{\partial R_1} = c_1 - ac_3\ R_1^{-a-1}\ R_2^{-b} = 0,\ \frac{\partial C}{\partial R_2} = c_2 - bc_3\ R_1^{-a}\ R_2^{-b-1} = 0.$$

It follows

$$R_1 = \frac{a}{c_1}\left(\left(\frac{c_1}{a}\right)^a \left(\frac{c_2}{b}\right)^b c_3\right)^{\frac{1}{a+b+1}}, \quad R_2 = \frac{b}{c_2}\left(\left(\frac{c_1}{a}\right)^a \left(\frac{c_2}{b}\right)^b c_3\right)^{\frac{1}{a+b+1}}.$$

The conditions $0 < R_i \le 1$ fix the parameters a, b, c_1, c_2, c_3. Moreover,

$$\frac{\partial^2 C}{\partial R_1^2} = a(a+1)\ c_3\ R_1^{-a-2}\ R_2^{-b},\ \frac{\partial^2 C}{\partial R_2^2} = b(b+1)\ c_3\ R_1^{-a}\ R_2^{-b-2}$$

$$\frac{\partial^2 C}{\partial R_1 \partial R_2} = ab\ c_3\ R_1^{-a-1}\ R_2^{-b-1}.$$

Since

$$\frac{\partial^2 C}{\partial R_1^2} > 0, \quad \frac{\partial^2 C}{\partial R_1^2}\frac{\partial^2 C}{\partial R_2^2} - \left(\frac{\partial^2 C}{\partial R_1 \partial R_2}\right)^2 > 0,$$

the cost $C(R)$ is a strict convex function in Euclidean sense. In this way the critical point is a minimum point and

$$\min\ C(R) = \left(\left(\frac{c_1}{a}\right)^a \left(\frac{c_2}{b}\right)^b c_3\right)^{\frac{1}{a+b+1}} (a+b+1).$$

Now let us formulate the dual program:

$$\sup V(\delta) = \left(\frac{c_1}{\delta_1}\right)^{\delta_1} \left(\frac{c_2}{\delta_2}\right)^{\delta_2} \left(\frac{c_3}{\delta_{31}}\right)^{\delta_3}$$

subject to

$$\delta_1, \delta_2, \delta_3 \ge 0,\ \delta_1 + \delta_2 + \delta_3 = 1,\ \delta_1 - a\delta_3 = 0,\ \delta_2 - b\delta_3 = 0.$$

We find

$$\delta_1 = \frac{a}{a+b+1},\ \delta_2 = \frac{b}{a+b+1},\ \delta_3 = \frac{1}{a+b+1}$$

and

$$\sup V(\delta) = \max V(\delta) = \min\ C(R) = M.$$

The minimum point of $C(R)$ satisfies also the system

$$c_1 R_1 = \frac{a}{a+b+1} M,\ c_2 R_2 = \frac{b}{a+b+1} M,\ c_3\ R_1^{-a}\ R_2^{-b} = \frac{1}{a+b+1} M.$$

5. Conclusions

In this paper a system reliability optimization problem through reliability allocation at the component level was examined using geometrical concepts. The problem was approached as a nonlinear programming problem with suitable objective functions and an active constraint as a Tzitzeica hypersurface. The advantage of our models is that the used mathematical technology can be applied to any system with high complexities.

Acknowledgements.

The work was partially supported by Academy of Romanian Scientists, and by University Politehnica of Bucharest.The paper was presented at International Conference on Applied Mathematics and Numerical Methods (ICAMNM), University of Craiova, April 14-16, 140 2016, Craiova, Romania.

References

J.-F. Auby, N. Brinzei, *Systems Dependability Assessment Modeling with Graphs and Finite State Automata*, John Wiley and Sons, Inc., 2015.

O. Calin, C. Udrişte, *Geometric Modeling in Probability and Statistics*, Springer, 2014.

K. Dohmen, *Inclusion-exclusion and network reliability*, The Electronic Journal of Combinatorics, Research Paper R36, 5 (1998), 1 - 8.

W. Kuo, V. R. Prasad, *An annotated overview of system-reliability optimization*, IEEE Transactions on Reliability, 49 (2000), 176 - 187.

P. B. Lakey, A. M. Neufelder, *System and Software Reliability Assurance Notebook*, Rome Laboratory, 1996, Rome NY, 6.1 - 6.24.

A Mettas, *Reliability allocation and optimization for complex systems*, Proceedings Annual Reliability and Maintainability Symposium, Los Angeles, CA, January 2000, 216-221.

C. Udrişte, E. Tănăsescu, *Minima and Maxima of Real Functions of Real Variables* (in Romanian), Technical Editorial House, Bucharest, 1980.

C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publisheres, 1994.