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Multi Dimensional Continuous Shearlet Transform and Multi Scale Wavelet Shrinkage with Integro Differential Equations

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Abstract

The aim of this paper is to extend the results of G. Liu et al. [16]. The traditional n-dimensional continuous wavelet transform used in [16] does not provide the information about the geometry of the set of singularities. These motivate us to consider a generalized wavelet transform associated with more general dilation groups. This generalized transform is known as continuous shearlet transform. Firstly, we relate 2-dimensional continuous shearlet transform to the sum of the smoothed partial derivative operators. Finally, this particular transform has been explained as a weighted average of pseudo-differential equations.2010 AMS Mathematics Subject Classification: 68U10, 47G20

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1. Introduction

During the last years, there has been growing interest in studying the wavelets and found that wavelet theory is important on several areas of pure and applied mathematics. It gave the understanding of many problems in various sciences, engineering and other disciplines and it includes, among its notable successes, the wavelet base digital fingerprint image compression standard adopted by the FBI in 1993 and JPEG2000, the current standard for image compression. It provided a very suitable vehicle for the analysis of the core linear operators of harmonic analysis and partial differential equations (Calderon – Zygmund theory). Given an image, the generic problems of image processing are compression, noise reduction, feature extraction and object recognition of which wavelet shrinkage plays an important role.

It should be noted that a very interesting thing about wavelet shrinkage ([1], [7-9], [11-12], [17], [19]) is that it can be motivated from very different fields of mathematics, namely partial differential equations, the calculus of variations, harmonic analysis or statistics, which, conversely gives a great impetus to the applications of wavelet to many engineering fields. Of these, Donoho et al. analyzed wavelet shrinkage methods in the context of minimax estimation and showed that wavelet shrinkage generates asymptotically optimal estimates or noisy data that outperform any linear estimator [7-9].

It is notable that the continuous wavelet transform has many advantage over the discrete wavelet transform, such as the weaker limited condition for the generate function, the arbitrary choice of the scale and translation, especially the wide application in the fields of pattern recognition, feature extraction and detection.

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Using the idea of understanding wavelets as smoothed derivative operators, Didas S and Weickert J [13] describe one dimensional wavelet shrinkage as approximation to a novel integro-differential evolution equation and compare the corresponding energy functional that uses both smoothed derivative operators within the penalties and integration over all scales, with the classical regularization ones. In [16] Liu, X. Feng and Min Li extend the results of [13] to n-dimensional $(n \geq 2)$ case. In one dimension, there are wavelets not arising from an MRA. Also, wavelets can be defined by replacing dyadic dilations with dilations by $r > 1$, where r need not be an integer. In the situation of non-dyadic dilations, the construction of the orthonormal bases associated with the wavelet may require more than one generator, namely, if $r = \frac{p}{q} > 1$ and p, q are relatively prime, then $p - q$ generators are needed. In order to avoid multiple wavelet generators in higher dimension we restrict our attention to $n \times n$ integer matrices u with $|\det u| = 2$ where the dilation set $\{2^j : j \in Z\}$ is replaced by $\{u^j : j \in Z\}$. The n-dimensional continuous wavelet transform used in [16] is able to describe the local regularity of functions and distribution and detect the location of singularity points through it decay at fine scales, it does not provide additional information about geometry of the set of singularities. Several constructions have been introduced, starting with the wedgelets [6] and ridgelets [3]. Among the most successful constructions proposed in the literature, the curvelets [4] and shearlets [14] achieve this additional flexibility by defining a collection of analyzing functions ranging not only over various scales and locations, like traditional wavelets, but also over various orientations and with highly anisotropic supports.

All of these motivate us to extend the results of [16]. In this paper we have considered wavelet transform associated with more general dilation groups. This generalized transform is called the continuous shearlet transform. Firstly, we relate a n-dimensional continuous mother wavelet $\varphi(x)$ with a fast decay and vanishing moments to the sum of the order partial derivative of a group function $\theta^k(x)(|k| = d)$ with fast decay, which also makes shearlet transform equal to the sum of the smoothed partial derivative operators. Finally, n-dimensional continuous shearlet transform can be explained as a weighted average of pseudo-differential equations.

2. Notations and n-Dimensional Fourier Transform with Some Properties

We write $x = (x_1, x_2, \ldots, x_n), k = (k_1, k_2, \ldots, k_n) \in N_0^n, N_0 = N \cup \{0\}$, let $|x| = \sum_{i=1}^n |x_i|$, $|k| = \sum_{i=1}^n k_i, x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$. For $y = (y_1, y_2, \ldots, y_n) \in R^n, x.y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$ and the partial derivative operators be $\frac{\partial^k}{\partial x^k} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$.

For $1 \leq p < \infty$, $n \in N$, let $L^p(R^n) = \{f: R^n \to R^n; \int_{R^n} |f(x)|^p \, dx < \infty\}$.

The space $L^2(R^n)$ is the Hilbert space of all square (Lebesgue) integrable functions endowed with inner product $< f, g > = \int_{R^n} f \bar{g}$. The Fourier transform is the unitary operator that maps $f \in L^2(R^n)$ into the function $\hat{f}$ defined by

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{-2\pi i \xi x} \, dx$$

when $f \in L^1(R^n) \cap L^2(R^n)$ and by the appropriate limit for the general $f \in L^2(R^n)$. The function $\hat{f}$ is also square integrable. Indeed, Fourier transforms maps $L^2(R^n)$ one-to-one onto itself and the inverse Fourier transform is defined by

$$\check{g}(x) = \int_{R^n} g(\xi) \, e^{2\pi i x \xi} \, d\xi.$$

The convolution of two functions $f, g \in L^1(R^n)$ is defined as for all $x \in R^n$

$$(f * g)(x) = \int_{R^n} f(x - \tau) g(\tau) d\tau.$$

The following properties of Fourier transform will be used throughout the paper.

1. Convolution. For $f, g \in L^1(R^n)$, we have

$$(\widehat{f * g})(\xi) = \hat{f}(\xi).\hat{g}(\xi).$$

2. Time partial derivatives. For $f \in L^1(R^n)$ and $x^k f \in L^1(R^n)$, we have

$$\frac{\partial^k f(\xi)}{\partial \xi^k} = \left((-ix)^k f(x)\right)(\xi).$$

3. Frequency partial derivatives. For $f \in L^1(R^n)$ and $\frac{\partial^k f(x)}{\partial x^k} \in L^1(R^n), 1 \leq k \leq n$, we have

$$\left(\frac{\partial^k \hat{f}(x)}{\partial x^k}\right)(\xi) = (i\xi)^k \hat{f}(\xi).$$

4. Scaling and translation property. For $f \in L^1(R^n), s > 0$ and $t^* \in R^n$, we have

$$f\left(\frac{x - t^*}{s}\right) = \exp\,(-it^*.\xi)\,|s|^n \hat{f}\,(s\xi).$$

3. Continuous Wavelets on R^n

In this section, we will study the continuous wavelets and continuous wavelet transform on R^n.Many of the concepts of one dimension can be extend to n −dimensions $(n \in N, n > 1)$ with Z −translations replaced by Z^n translations and the dilation set $\{2^j : j \in Z\}$ replaced by $\{u^j : j \in Z\}$, where u is an $n \times n$ real matrix, each of whose eigenvalues has magnitude larger than one. Like the Fourier transform, the wavelet transform has a discrete and a continuous version. For the continuous wavelet transform, one has weaker conditions, especially the orthogonality not necessary for an invertible continuous wavelet transform. So, it is convenient to focus our attention on Parseval frame (PF) wavelets rather than orthonormal wavelets. Thus, given the matrix$u = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, for $n = 2$ and $\varphi \in L^2(R^n)$ the wavelet system

$$\varphi_{j,k}(x) = \left(D_u^j T_k \varphi\right)(x)$$
$$= |\det u|^{\frac{j}{2}} \varphi\left(u^j x - k\right), j \in Z, k \in Z^n, \tag{3.1}$$

is a PF. That is,

$$\sum_{j \in Z} \Big(\sum_{k \in Z^n} | < f, D_u^j T_k \varphi > |^2 = ||f||^2_{L^2(R^2)} \Big)$$

for all $f \in L^2(R^n)$.

In general, the construction of two-dimensional wavelets is significantly more complicated than the one-dimensional case. In particular, it is not known whether there exists continuous compactly supported orthonormal wavelet analogous of the one-dimensional Daubechies wavelets associated with the dilation quincunx matrix$q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. However, it turns out that a rather simple change in the definition of the dilation (3.1) produces much simpler constructions of Haar type wavelets in dimension two.

In [15] K. Guo, W-Q, Lim, D. Labate, G. Weiss and E. Wilson, introduced the notion of wavelet system with composite dilations, which are the systems of the form

$$\{D_a D_b T_k \varphi : j \in Z, a \in A, b \in B, k \in Z^2\},$$

where B is a group of matrices with determinant of absolute value 1 and A is a group of expanding matrices, in the sense that all eigenvalues have magnitude larger than one. Many different groups of Bdilations have been considered in the literature, such as the crystallographic and shear group, which are associated with very different properties. One special benefit of this framework is the ability to produce wavelet like systems with geometric properties going far beyond traditional wavelets. For example, one can construct waveforms whose supports are highly anisotropic and that range not only over various orientations, making these functions particularly useful in image processing applications.

Continuous Wavelet Transform: Let the general linear group $GL(n, R)$ of $n \times n$ invertible real matrices acts on R by linear transformations. The semidirect product G of $GL(n, R)$ and $R^n$is called the general affine group on R^n, since each invertible affine map on R^n has the form $(a, t).x = a(x + t) = ax + at$ for a unique $(a, t) \in G$. Thus, the group law $(a, t).(b, s) = (ab, b^{-1}t + s)$ for G corresponds to the composition of the associated affine maps and $(a, t)^{-1} = (a^{-1}, -a^{-1}t)$. A unitary representation τ of G acting on $L^2(R^n)$ is defined by

$$\left(\tau_{(a,t)}\varphi\right)(x) = |\det a|^{-\frac{1}{2}}\varphi((a, t)^{-1}.x)$$

$$= |\det a|^{-\frac{1}{2}}\varphi(a^{-1}x - t) = \varphi_{a,t}(x),$$

and

$$(\widehat{\tau_{(a,t)}\ \varphi})(\xi) = |\det a|^{\frac{1}{2}}\hat{\varphi}(a^*\xi)e^{-2\pi i\xi.at}.$$

For $\varphi \in L^2(R^n)$, the continuous wavelet transform we associated with φ and G is defined by

$$\left(W_{\varphi f}\right)(a, t) = < f, \varphi_{(a,t)} >$$
$$= |\det a|^{-\frac{1}{2}} \int_{R^n} f(x)\, \overline{\varphi(a^{-1}x - t)}dx,$$

and it maps $f \in L^2(R^n)$ into a space of functions on G. For D a closed subgroup of $GL(n, R), H = \{(a, t): a \in D, t \in R^n\}$ is a closed subgroup of G and the left Haar measures on H are the product measures $d\lambda(a, t) = d\mu(a)dt$, where μ is a left Haar measure on D. It should be mentioned that, we seek conditions on D and μ for which restricting $W_{\varphi f}$to H gives an isometry from $L^2(R^n)$ into $L^2(H, d\lambda)$ and we have

$$f = \int_H < f, \tau_{(a,t)}\varphi > \tau_{(a,t)}\mu d\lambda(a, t),$$

for each $f \in L^2(R^n)$. As given in [2], this holds if and only if

$$\int_D \left|\hat{\varphi}\left(a^{*\xi}\right)\right|^2 d\mu(a) = 1, \text{for } a.e, \xi \in R^n, \tag{3.2}$$

in which case μ is a continuous wavelet with respect to D. Particularly if $D = \{aI_n : a > 0\}$, where I_n is the $n \times n$ identity matrix and $d\mu(aI_n) = \frac{da}{a}$, the condition (3.2) reduces to the classical Colderon condition

$$\int_0^\infty |\hat{\varphi}(a\xi)|^2 \frac{da}{a} = 1, \text{for } a.e, \xi \in R^n.$$

For $G = D \times R^n$, the continuous wavelet transform of $f \in L^2(R^n)$ is defined as

$$\begin{aligned}(W_{\varphi f})(a,t) &= (W_{\varphi f})(aI_n,t) \\ &= a^{-\frac{n}{2}}\int_{R^n} f(x)\overline{\varphi\big(a^{-1}(x-t)\big)}dx = (\tilde{\varphi}_a * f)(x),\end{aligned} \tag{3.3}$$

where

$$\varphi_a(x) = a^{-n/2}\,\varphi\left(\frac{x}{a}\right), \tilde{\varphi}(x) = \overline{\varphi(-x)}, x \in R^n, a > 0.$$

Here we used the function $\varphi(a^{-1}(x-t))$ instead of $\varphi(a^{-1}x - t)$ because it simplifies the problem of estimating the asymptotic decay properties of the continuous wavelet transform. A fundamental property of this transform is its ability to characterize the local regularity of functions.

4. Wavelet Shrinkage

Wavelet shrinkage became popular the work of Donoho and Johnstone [10]. The general idea is to transform the data to a representation that allows reducing noise in a straight forward way, namely by diminishing the modulus of the wavelet coefficients. Especially the low computational complexity of the wavelet transform has made such approaches highly interesting for signal and image processing applications. A typically nonlinear – shrinkage function $S: R \to R$ is applied to the wavelet transform$W_\varphi f$. Some assumptions on S are $x > 0 \Rightarrow S(x) \geq 0, S(-x) = -S(x)$and $|S(x)| \leq |(x)|$ for all $x \in R$.usually S depends on a parameter λ^* which determines the amount of shrinkage. This parameter is omitted here to simplify the notations.

It is significant to mentioned here that the wavelet transform $W_\varphi f$ is an isometric map from $L^2(R^n)$to $L^2(H, d\lambda)$. Thus on the subspace given by the image of W_φ, the adjoint operator W_φ^* is the inverse ofW_φ.

Thus, a function can be reconstructed from its wavelet transform by means of the formula

$$\begin{aligned}u(x) = W_\varphi^*(SoW_\varphi f) &= \int_0^\infty \left(\varphi_a * S\left(W_\varphi f(.,a)\right)\right)\frac{da}{a^{n+1}} \\ &= \int_0^\infty \frac{da}{a^{n+1}}\int_{R^n}(W_\varphi f)\frac{1}{a^{\frac{n}{2}}}\overline{\varphi\big(a^{-1}(x-\xi)\big)}\,d\xi \\ &= \int_0^\infty (\varphi * S\,(\varphi_a * f))\,\frac{da}{a^{n+1}}.\end{aligned}$$

The continuous wavelet transform (3.3) is able to describe the local regularity of functions and distribution and detect the location of singularity points through its decay at fine scale it does not provide additional information about the geometry of the set of singularities. In order to achieve this addition capability, we have to consider wavelet transforms associated with more general dilation group.

Let M be the subgroup of $GL(2,R)$ of the matrices

$$\left\{m_{a,s} = \begin{pmatrix} a & -a^{\frac{1}{2}}s \\ 0 & a^{\frac{1}{2}}\end{pmatrix} : a > 0, s \in R\right\}.$$

and let us consider the corresponding generalized continuous wavelet transform

$$(W_\varphi f)(a,s,t) = (W_\varphi f)(m_{a,s}t)$$

$$= a^{-\frac{3}{4}} \int_{R^n} f(x)\overline{\varphi\left(m_{a,s}^{-1}(x-t)\right)}\,dx$$

$$= (\tilde{\varphi}_{a,s} * f)$$

where $a > 0, s \in R, t \in R^2, \varphi_{a,s}(x) = a^{-\frac{3}{4}}\left(\frac{x}{m_{a,s}}\right)$ and $\tilde{\varphi}(x) = \overline{\varphi(-x)}$.

We can verify the factorization $m_{a,s} = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{\frac{1}{2}} \end{pmatrix}$; it shows that $m_{a,s}$ the product of an anisotropic dilation matrix and a shear matrix. Thus, the analyzing function $\varphi_{a,s,t} = a^{-3/4}\,\varphi(m_{a,s}^{-1}(x-t))$ associated with this transform ranges over various scalers, orientations, and locations controlled by the variables a, s, t respectively. The transform $(W_{\varphi f})(a,s,t)$ is called the continuous shearlet transform of f. This transform is able to detect not only the location of singularity points through its decay at fine scales but also the geometric information of the singularity set.

In a similar manner a function can be reconstructed from its shearlet transform by means of the formula

$$u(x) = W_\varphi^*(SoW_\varphi f) = \int_0^\infty \left(\varphi_{a,s} * S\left(W_\varphi f(.,a,s)\right)\right)\frac{da}{a^{\frac{5}{2}}}$$

$$= \int_0^\infty \frac{da}{a^{5/2}} \int_{R^2} (W_\varphi f) a^{-\frac{3}{4}}\,\varphi\,(m_{a,s}^{-1}(x-\xi)\;d\xi \qquad (4.1)$$

$$= \int_0^\infty \left(\varphi_{a,s} * S(\tilde{\varphi}_{a,s} * f)\right)\frac{da}{a^{5/2}}\,.$$

5. Continuous Shearlet Transform and the Sum of Smoothed Partial Derivative Operators

Definition 5.1 [18]. The mother wavelet $\varphi(x)$ has fast decay if for any exponent $r \in N$ there exists a constant C_r such that for all $x \in R^n$.

$$|\varphi(x)| \leq \frac{C_r}{1+|x|^r}.$$

Definition 5.2 [18]. The mother wavelet $\varphi(x)$ has d (a finite number $d \in N$) order vanishing moments if $\int_{R^n} x^k \varphi(x)dx = 0$ for $k = (k_1, \dots, k_n) \in N^n, 0 \leq |k| < d.$

Lemma 5.1 [16]. A function f is bounded and p times continuously partial differentiable with bounded partial derivatives if

$$\int_{R^n} |\hat{f}(\xi)|(1+|\xi|^p)\,d\xi < \infty.$$

If the mother wavelet $\varphi(x)$ satisfies Definitions 5.1 and 5.2 we get the following theorem about the shearlet transform and the smoothed partial operators.

Theorem 5.1. Wavelet $\varphi(x)$ with a fast decay has a vanishing moment if and only if there exist a group of functions $\theta^k(x)(|k| = d)$ with fast decay such that

$$\varphi(x) = (-1)^d \sum_{|k|=d} \frac{\partial^k \theta^k(x)}{\partial x^k}, x \in R^2. \quad (5.1)$$

As a consequence,

$$W\,f\,(a,s,t) = a^{3d/2} \sum_{|k|=d} \frac{\partial^k}{\partial\, t^k}\left(f * \tilde{\theta}^k_{a,s}\right)(t)$$

with

$$\tilde{\theta}^k_{a,s}(x) = a^{-3d/4}\theta^k\left(\frac{-x}{m_{a,s}}\right).$$

It has no more than vanishing moments if and only if

$$\sum_{|k|=d} (-i)^d C^k_d k! \int_{R^n} \theta^k(x)dx \neq 0. \; |k| = k_1 + k_2 < d.$$

Proof. The proof of the theorem follows on the lines of proof of [(16], Thm. A).

6. Two Dimensional Wavelet Shrinkage and Evolution Equation

In this section we shall study the relationship between 2 −dimensional wavelet shrinkage and an integro-differential evolution equation involving the data on a continuous spectrum of scales. Using (5.1), we have

$$\widetilde{\varphi_{a,s}} * f = a^{3d/2} \sum_{|k|=d} \frac{\partial^k}{\partial\, x^k}\left(\tilde{\theta}^k_{a,s} * f\right) = a^{3d/2}\left(\sum_{|k|=d} \frac{\partial^k_{a,s}}{\partial\, x^k} * f\right)$$

and

$$\varphi_{a,s} * f = \left(-a^{\frac{3}{2}}\right)^d \sum_{|k|=d} \frac{\partial^k}{\partial\, x^k}\left(\widetilde{\theta^k_{a,s}} * f\right) = \left(-a^{\frac{3}{2}}\right)^d \left(\sum_{|k|=d} \frac{\partial^k \widetilde{\theta^k}}{\partial\, x^k} * f\right).$$

So, for $n = 2$, shearlet translation $W_\varphi f(m_{a,s}\, f)$ and the adjoint transform $W^*_\varphi\, f(m_{a,s}\, t)$ are expressed as

$$W_\varphi f\,(a,s,t) = W_\varphi\, f(m_{a,s}\, t)$$

$$= \tilde{\varphi}_{a,s} * f = a^{3d/2} \sum_{|k|=d} \frac{\partial^k}{\partial\, t^k}\left(\tilde{\theta}_{a,s} * f\right), \quad (6.1)$$

$$W^*_\varphi\, f(a,s,t) = \int_0^\infty (\varphi_{a,s} * f)\frac{da}{a^{5/2}} = \int_0^\infty (-a^{\frac{3}{2}} \sum_{|k|=d} \frac{\partial^k}{\partial\, t^k}\,(\theta^k_{a,s} * f)\frac{da}{a^{5/2}} \quad (6.2)$$

2 −Dimensional shearlet transform $W_\varphi\, f$ is equivalent to taking a sum of a smoothed partial derivative with an additional factor $a^{3d/2}$, and the adjoint transform $W^*_\varphi\, f$ is an additional integration over all scales $a^{3/2}$.Now choose an unitary function g such that [13]

$$S(x) = x - g(|x|^2)x.$$

Therefore, from (4.1), we observed that this expression is equivalent to

$$u(x) - f(x) = -\int_0^\infty \varphi_{a,s} * \left(g\left(\left|\tilde{\varphi}_{a,s} * f\right|^2\right)(\tilde{\varphi}_{a,s} * f)\right)\frac{da}{a^{\frac{5}{2}}},$$

using (6.1) and (6.2) we get the following equivalent formulation of wavelet shrinkage as integro – differential equation:

$$u - f = (-1)^{d+1}\int_0^\infty \sum_{|k|=d} \frac{\partial^k \theta_{a,s}^k}{\partial x^k} \tag{6.3}$$

$$\neq \left(g\left(\left|a^{\frac{3d}{2}}\sum_{|k|=d}\frac{\partial^k \theta_{a,s}^k}{\partial x^k} * f\right|^2\right)\left(\sum_{|k|=d}\frac{\partial^k \theta_{a,s}^k}{\partial\, x^k} * f\right)\right)\frac{da}{a^{5/2}}.$$

Similar to the discussion in [13], (6.3) can be understood as a single step time explicit approximation to the evolution equation:

$$\partial_t\, u = (-1)^{d+1}\int_0^\infty a^{3d}\sum_{|k|=d}\frac{\partial^k \theta_{a,s}^k}{\partial\, x\hat{}k} \tag{6.4}$$

$$* \left(g\left(\left|a^{\frac{3d}{2}}\sum_{|k|=d}\frac{\partial^k \theta_{a,s}^k}{\partial\, x^k} * f\right|^2\right)\cdot\left(\sum_{|k|=d}\frac{\partial^k \theta_{a,s}^k}{\partial\, x^k} * f\right)\right)\frac{da}{a^{\frac{5}{2}}}$$

If we compare (6.4) to the higher order nonlinear diffusion equation for the multivariate function

$$\partial_t u = (-1)^{d+1}\left(\sum_{|k|=d}\frac{\partial^k}{\partial\, x^k}\, g\left(\left|a^{\frac{3d}{2}}\sum_{|k|=d}\frac{\partial^k f}{\partial\, x^k}\right|^2\right)\sum_{|k|=d}\frac{\partial^k f}{\partial\, x^k}\right).$$

We find two main differences,

1. All appearing partial operators in the equation are pre-smoothed by convolution with scaled and mirrored versions of a set of kernel functionsθ^k.

2. The right hand side is not only considered at one single scale but there is integration over all scales with additional weight factors.

7. Conclusions and Applications

The aim of the present paper is to extend the results of [16]. Since the traditional n −dimensional continuous wavelet, transform used in [16] does not provide the information about the geometry of the set of singularities, in order to achieve this additional flexibility in this paper we consider the generalized wavelet transform namely continuous shearlet transform and showed that it is equal to the sum of the smoothed partial derivative operators. Moreover, 2 −dimensional continuous shearlet transform can be explained as a weighted average of pseudo-differential equations too.

In the last few years, we have seen an explosion of activity in machine learning, data analysis, and search, implying that similar ideas and concepts, inspired by signal processing weight carry as much power in the context of the orchestration of massive high dimensional data sets.

This digital data, medical records, music, sensor data, financial data etc., can be structured into geometries that result in new organizations of language. As an application, this work is useful in the oil exploration and mining industry, in which one needs to decide where to drill or mine to greatest advantage for finding oil, gas, copper or other minerals. In medical diagnostics, important information can be learned from the analysis of data obtained from radiological, histological, chemical tests and this is important for arriving at an early detection of potentially dangerous tumors and other pathologies (see [5]).

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