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On Lattice Properties of GV-Semi Groups

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Abstract

The main aim of the paper is to characterize a GV-semi group S whose lattice of GV-sub semi groups is 0-modular or 0-distributive. First, a GV-semi group with 0-modular GV-sub semi group's lattice is considered. Then we investigated a GV-semi group with 0-distributive GV-sub semi group's lattice. Finally, the results on a completely regular semi group with 0-modular or 0-distributive completely regular sub semi groups lattice be obtained.

Keywords: GV-semi group, GV-sub semi group lattice, 0-modular, left (right) zero band

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1. Introduction and preliminaries

Semi group theory concerning the sub semi group lattices of semi groups has been investigated for the past quarter of a century, especially by prof Shevrin and his colleagues. Under the influence of them, many algebraists have begun today more attention to the subject. Most of these results can be found in [7].Using the similar way in [7], inverse semi groups for which the inverse sub semi groups lattice is distributive, modular were determined in [3] by Ershova. Moreover, eventually inverse semi groups with their eventually inverse sub semi groups lattice have been considered by Tian,Z,J in[9],[10],[11]. Considering completely regular semi groups and GV-semi groups with their respective type sub semi groups lattice have not been studied up to now, the authors apply some approaches in [7] to character GV-semi groups whose lattice of GV-sub semi groups is 0-modular or 0-distributive. Then the corresponding results on completely regular semi groups can be got similarly.

A semi group S is called eventually regular if for every element a of S there exists $m \in Z^+$ (the set of positive integers) such that a^m is regular. We refer to $r(a)$ as the least positive integer m such that a^m is regular. If every regular element of an eventually regular semi group S is completely regular, then S is called GV-semi group. For a completely regular semi group S, every regular element a of S exists and only exists an inverse of a which commutes with a. We usually denote the unique inverse of a by a^{-1}. Thus, a sub semi group A of S is a completely regular sub semi group of S if $a^{-1} \in A$ for any $a \in A$. We get every regular element a of a GV-semi group S which is seen as the generalization of a completely regular semi group also exists and only exists an inverse of a which commutes with a.

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Therefore, we refer to a^{-1} as the unique inverse of a. Let S be a GV-semi group and A is a sub semi group of S. We say that A is a GV-sub semi group of S if for any $a \in A \cap \mathrm{Reg}S$, $a^{-1} \in A$. Obviously, A is a GV-sub semi group of S if and only if $A \cap \mathrm{Reg}S = \mathrm{Reg}A$.

In the proof of the following lemma, The relations L^*, R^* and H^* on a semi group S are generalization of the familiar Greens relations L, R and H as in [4].

Lemma 1.1 S is a GV-semi group if and only if S is an eventually regular semi group and every regular element a of S has a unique inverse which commutes with a.

Proof. Let S be a GV-semi group, then S is eventually regular. For any $a \in \mathrm{Reg}S = GrS$, there exists $x \in S$ such that $a = axa$; $ax = xa$ .Hence $y = xax \in V(a)$; $ay = axax = xaxa = ya$ .Now, we suppose there exists $z \in V(a)$ such that $az = za$, therefore $z = zaz = z^2a = z^2ayaya = z^2a^3y^2 = azazay^2 = ay^2 = y$;thus a has a unique inverse commuting with a.

Conversely, let S be an eventually regular semi group, then there exists $m = r(a)$ such that $a^m \in \mathrm{Reg}S$ for any $a \in S$. Put $x \in V(a^m)$ such that $a^m x = xa^m \in Es$. Notice $a^m x = a^m x$, $(a^m x)a^m = a^m$, hence $a^m R a^m x$, thus $aR^* a^m x$ .Symmetrically, we have $aL^* a^m x$. Thus $aH^* a^m x$, that is, every H^*-class of S contains one idempotent. Consequently, we get S is a GV-semi group.

Suppose S is a GV-semi group and A a subset of S. We will denote by $\langle A \rangle$ the sub semi group of S generated by A, by $GV\langle A \rangle$ the GV-sub semi group of S generated by A, and by $SubGVS$ the set of all GV-sub semi groups(including the empty set) of S. It is obvious that the set $SubGVS$ forms a complete lattice with respect to intersection denoted by $\wedge$ and union denoted by $\vee$, where $GV\langle A, B \rangle$ refers to the GV-sub semi group of S generated by the union of subsets A and B of S. Let X be a subset of S, we denote by $\langle X \rangle^{-1}$ as the set $\{x^{-1} : x \in \langle X \rangle \cap \mathrm{Reg}S\}$.

For a completely regular semi group S, $CR\langle A \rangle$ denotes the completely regular sub semi group of S generated by the subset of A of S and $CR\langle A, B \rangle$ denotes the completely regular sub semi group of S generated by the union of subsets A, B of S. Obviously, we have the set $SubCRS$ which refers to all completely regular sub semi groups (including the empty set) of S also forms a complete lattice with the operations $\vee$ and $\wedge$, where $A \wedge B = A \cap B$, $A \vee B = CR\langle A, B \rangle$.

A semi group S is called an epi group, if for any $a \in S$, there exists $n \in Z^+$ such that a^n lies in a subgroup of S. If an epi group S only has an idempotent, then it is named unipotent epi group. Given $e \in Es$, G_e denotes the maximal subgroup of a semi group S containing e and we put

$$K_E = \{x \in S : x^n \in G_e \text{ for some } n \in Z^+\},$$

Then we define the relation $\kappa = \bigcup_{e \in E_S}(K_e \times K_e)$. The relation κ is an equivalent relation on an epi group and $K_e(e \in E_S)$ is called a unipotency class of S. Moreover, if S is a GV-semi group, then $K_e = \mathrm{H}_e^*$ for any $e \in E_S$ which is a sub semi group of S. A band S is named a left (right) zero band if $ab = a(ab = b)$.

The following lemma will be used several times in the sequel.

Lemma 1.2 ([2]) Let S be a GV-semi group, then S is a band of unipotentepi groups $K_e(e \in E_S)$ if and only if the relation $\kappa(\mathrm{H}^*)$ is congruence on S.

A lattice$(L, \wedge, \vee)$ with zero is called 0-distributive, if for any $a, b, c \in L$, $a \wedge b = a \wedge c = 0$. A lattice$(L, \wedge, \vee)$ is named 0-modular, if for any $a, b, c \in L$, $a \wedge b = 0, c \leq a$ implies $a \wedge (b \wedge c) = c$. Although distributivity is stronger than modularity, there is no implicative relation between 0-distributive and 0 - modular.

For the terminology and notation which is not given here, the reader is referred to [1],[4],[7].

2. The case of 0-modularity

A GV-semi group S is called a GU-band of GV-semi groups $S_\alpha(\alpha \in Y)$ if $S = \bigcup_{\alpha \in Y} S_\alpha$ is a band of S_α and $xy, yx \in GV\langle x \rangle \cup GV\langle y \rangle$ for any $x \in S_\alpha, y \in S_\beta$ with $\alpha \neq \beta$. Similarly, a completely regular semi group S is called a CU-band of completely regular semi groups $S_\alpha(\alpha \in Y)$ if $S = \bigcup_{\alpha \in Y} S_\alpha$ is a band of S_α and $xy, yx \in CR\langle x \rangle \cup CR\langle y \rangle$ for any $x \in S_\alpha, y \in S_\beta$ with $\alpha \in \beta$.

Lemma 2.1 Let S be a GV-semi group. If $SubGVS$ is 0-modular, then $\{e, f\}$ is a left (right) zero band or chain for any $e, f \in E_S$ with $e \neq f$.

Proof. First, we put $A = GV\langle ef, e\rangle, B = GV\langle f\rangle = \{f\}, C = GV\langle e\rangle = \{e\}$ for any $e, f \in E_S$ with $e \neq f$.

If $A \cap B = \phi$, Clearly $C \leq A$, the $GV\langle ef, e\rangle \cap GV\langle e, f\rangle = GV\langle ef, e\rangle = \{e\}$ by 0-modularity of $SubGVS$. Hence $GV\langle ef\rangle = e$, that is $ef = e$.

If $A \cap B \neq \phi$. It implies $f \in GV\langle ef, e\rangle$. Hence $GV\langle e, f\rangle \subseteq GV\langle ef, e\rangle$, and so $GV\langle e, f\rangle = GV\langle ef, e\rangle$ by $GV\langle ef, e\rangle \subseteq GV\langle e, f\rangle$. Next we show e is a leftidentity of $GV\langle ef, e\rangle$. It is obvious that e is a left identity of $\langle ef, e\rangle$. For any $b \in \langle ef, e\rangle^{-1}$, then there exists $a \in RegS \cap \langle ef, e\rangle$ such that $a^{-1} = b$ by the definition of $\langle ef, e\rangle^{-1}$. Hence $eb = ea^{-1} = ea^{-1}aa^{-1} = eaa^{-1}a^{-1} = a^{-1} = b$,

Thus e is also a left identity of $\langle ef, e\rangle^{-1}$. Now put $A_1 = \langle\langle ef, e\rangle, \langle ef, e\rangle^{-1}\rangle$. For any $a_1 \in A_1$, then there exist $x_1, x_2, \ldots, x_n \in \langle ef, e\rangle \cup \langle ef, e\rangle^{-1}$ with $n \in Z^+$ such that $a_1 = x_1x_2, \ldots x_n$. Notice that

$$ea_1 = ex_1x_2, \ldots x_n = x_1x_2, \ldots x_n = a_1 \text{ by } ex_1 = x_1.$$

Since e is a left identity of $\langle ef, e\rangle \cup \langle ef, e\rangle^{-1}$. It follows that e is a left identity of A_1. Next suppose any $b_1 \in A_1^{-1}$, there exists $a_2 \in \operatorname{Re} gS \cap A_1$ such that $a_2^{-1} = b_1$. Whence $eb_1 = ea_2^{-1} = ea_2^{-1}a_2a_2^{-1} = ea_2a_2^{-1}a_2^{-1} = b_{-1}$, that is, e is a left identity of A_1^{-1}. Similarly, we get e is a left identity of $\langle A_1 A_1^{-1}\rangle$. Repeating the procession, we can show e is a left identity of $GV\langle ef, e\rangle$. Therefore, e is a left identity of $GV\langle e, f\rangle = GV\langle ef, e\rangle$, and so $ef = f$.

By the above analysis, we get $\{e, f\}$ is a left(right) zero band or chain for any $e, f \in Es$ with $e \neq f$.

From the lemma 2.1, we have the auxiliary corollary easily.

Corollary 2.2 Let S be a GV-semi group whose $SubGVS$ is 0-modular, then E_S is a band and $\operatorname{Re} gS$ is a completely regular sub semi group of S.

Lemma 2.3 Let S be a GV-semi group. If $SubGVS$ is 0-modular, then

$GV(x, g) = GV(x) \cup g$ For any $x \in K_e, e, g \in Es$ with $e \neq g$.

Proof. From lemma 2.1, we know $\{e, g\}$ is a left (right) zero band or chain for any $e, g \in Es$ with $e \neq g$, so we first consider the case that $\{e, g\}$ is a left zero band.

The assertion of the lemma is trivial if $x = e$, hence assume $x \in K_e \setminus \{e\}$. Next we show that $gx, xg \in GV\langle x\rangle \cup g$ for any $x \in K_e \setminus \{e\}$. First, suppose that

$gx \notin GV\langle x\rangle \cup g$ and put $A = GV\langle gx, g\rangle$, $B = GV\langle x\rangle$, $C = GV\langle g\rangle = \{g\}$.

If $A \cap B = \phi$, Clearly $C \leq A$, hence $GV\langle gx, g\rangle \cap GV\langle x, g\rangle = \{g\}$. It is obvious that $gx \in GV\langle gx, g\rangle \cap GV\langle x, g\rangle = \{g\}$, which contradicts the hypothesis, and so $A \cap B \neq \phi$. Furthermore, we get $e \in GV\langle gx, g\rangle$ *by* $x \in K_e$. Clearly, g is a left identity of $\langle gx, g\rangle$. We can prove g is also a left identity of $GV\langle gx, g\rangle$ as the proof in lemma 2.1, hence $ge = e$. On the other hand, we get $ge = g$ since $\{e, g\}$ is a left zero band. Thus $ge = e = g$ which leads to a contradiction. Consequently, we have $gx \in GV\langle x\rangle \cup g$.

Next, suppose $xg \notin GV\langle x\rangle \cup g$ and put

$A' = GV\langle xg, g\rangle$, $B' = GV\langle x\rangle$, $C' = GV\langle g\rangle = \{g\}$. If $A' \cap B' = \phi$,

then $GV\langle xg, g\rangle \cap GV\langle x, g\rangle = \{g\}$ by 0-modularity of $SubGVS$ and $C' \leq A'$. That $xg \in GV\langle xg, g\rangle \cap GV\langle x, g\rangle = \{g\}$ is obvious, hence it contradicts the hypothesis, and so $A' \cap B' \neq \phi$. For any $m > 1, m \in Z^+$, we have

$(xg)^m = xgexg \ldots xg = xgx^{m-1}, (gx)^m = gexgx \ldots gx = gx^m$.

According to the above analysis, we have known $gx \in GV\langle x\rangle \cup g$, furthermore $(gx)^m \in GV\langle x\rangle \cup g$ for any $m > 1$ with $m \in Z^+$.

If $gx \in GV\langle x\rangle$, then $(gx)^{m-1} \in GV\langle x\rangle$, $(xg)^m = xgx^{m-1} = x(gx)^{m-1} \in GV\langle x\rangle$.

If $gx = g$, then $(xg)^m = x(gx)^{m-1} = xg$, hence xg is periodic. Analyzing the elements in $\langle xg, g\rangle$, we have all the elements containing in it are g, $(xg)^t$, $g(xg)^t = (gx)^t$ for any $t \in Z^+$. Therefore $GV\langle xg, g\rangle = \langle xg, g\rangle$ since g, $(xg)^t$ and $g(xg)^t = (gx)^t$ are all periodic. Notice $GV\langle xg, g\rangle \cap GV\langle x\rangle \neq \phi$ and $(gx)^t = g$ for any $t \in Z^+$, hence there exists $\dot{m} \in Z^+$ such that $(xg)^{\dot{m}} \in GV\langle x\rangle$. It follows there exists $m_1 \in Z^+$ such that $(xg)^{m_1} \in GV\langle x\rangle$ when $gx \in GV\langle x\rangle \cup g$. Let $r(x) = n$. Then $xg = xge = (xgx^{m_1-1})x^{t_1+1}(x^n)^{-1} \in GV\langle x\rangle$ by $xgx^{m-1} = (xg)^{m_1} \in GV\langle x\rangle$. Hence, it contradicts the hypothesis and so $xg \in GV\langle x\rangle \cup g$.

For any $a \in GV\langle x\rangle \subseteq K_e$, then $ag, ga \in GV\langle a\rangle \cup g \subseteq GV\langle x\rangle \cup g$ by the above analysis. Whence $GV\langle x\rangle \cup g$ is a subsemi group of S, and so $GV\langle x\rangle \cup g \in SubGVS$. It is obvious that $\langle x, g\rangle \subseteq GV\langle x\rangle \cup g$, whence $GV\langle x, g\rangle \subseteq GV\langle x\rangle \cup g$ and clearly $GV\langle x, g\rangle \supseteq GV\langle x\rangle \cup g$, thus $GV\langle x, g\rangle = GV\langle x\rangle \cup g$. That $GV\langle x, g\rangle = GV\langle x\rangle \cup g$ can be proved similarly.

By lemma 2.1 and lemma 2.3, we can prove the following lemma.

Lemma 2.4 Let S be a GV-semi group and $SubGVS$ is 0-modular, then

$xy, yx \in GV\langle x\rangle \cup GV\langle y\rangle$ for any $x \in K_e, y \in K_f$ with $e, f \in Es, e \neq f$.

Proof. First suppose there exist $a \in K_e, b \in K_f$ such that $ab \notin K_e \cup K_f$ with

$e, f \in E_S, e \neq f$. Then there exists $g \in E_S \setminus \{e, f\}$ such that $ab \in K_g$. Hence

$(ab)^n \in K_g, (ab)^n \notin K_e \cup K_f$ for any $n \in Z^+$. Put $A = GV\langle a, g\rangle$, $B = GV\langle b\rangle$, $C = GV\langle a\rangle$ By lemma 2.3, we get $A = GV\langle a, g\rangle = GV\langle a\rangle \cup g$. Hence

$A \cap B \neq \phi$ and clearly $C \leq A$, and so $GV\langle a, g\rangle \cap GV\langle a, b\rangle = GV\langle a\rangle$ by 0-modularity of $SubGVS$. Notice

$$GV\langle a, g\rangle \cap GV\langle a, b\rangle = GV\langle a, b\rangle \cap (GV\langle a\rangle \cup g) = GV\langle a\rangle \cup g,$$

Whence $GV\langle a\rangle \cup g = GV\langle a\rangle$, $i.e.\ g \in GV\langle a\rangle$ and so it leads to a contradiction.

Thus $K_e K_f \subseteq K_e \cup K_f$. Symmetrically, we can prove $K_f K_e \subseteq K_e \cup K_f$ for any $e, f \in Es, e \neq f$.

According to lemma 2.1, we have $ef = e$ or $ef = f$ for any $e, f \in Es$ with $e \neq f$. Next let $ef = e$ and assume there exist $a \in K_e, b \in K_f$ such that $ab \in K_f$. Hence $af = (ab)b^{r(b)-1}(b^{r(b)})^{-1} \in K_f G_f \subseteq G_f$ and since there exists $n \in Z^+$ with $n \geq r(a)$ such that $a^n \in G_e$, whence

$$a^n f = aa \cdots f^n = (af)^n, a^n = a^n e \text{ and } a^n ef = a^n f = (af)^n \in K_f.$$

Therefore, $K_e \cap K_f \neq \phi$ by $a^n ef = a^n e = a^n \in K_e$ which contradicts the fact

$K_e \cap K_f = \phi$ with $e \neq f$. Thus $K_e K_f \subseteq K_e$. Dually, we have $K_e K_f \subseteq K_f$ when $ef = f$ for any $e, f \in Es$ with $e \neq f$.

Suppose $K_e K_f \subseteq K_e$ for any $e, f \in Es$ with $e \neq f$ and put

$A = GV\langle xy, x\rangle, B = GV\langle y\rangle$ and $C = GV\langle x\rangle$ for any $x \in K_e, y \in K_f$. Obviously, $A \cap B = \phi$ and $C \leq A$, hence $GV\langle xy, x\rangle \cap GV\langle x, y\rangle = GV\langle xy, x\rangle = GV\langle x\rangle$

by the 0-modularity of $SubGVS$, and so $xy \in GV\langle x\rangle$. Assume $K_e K_f \subseteq K_f$ for any $e, f \in Es$ with $e \neq f$ and put $A = GV\langle xy, x\rangle$, $B = GV\langle y\rangle$ and $C = GV\langle x\rangle$ for any $x \in K_e$, $y \in K_f$. Clearly, $A \cap B = \phi$ and $C \leq A$, whence we get

$GV\langle xy, x\rangle \cap GV\langle x, y\rangle = GV\langle xy, y\rangle = GV\langle y\rangle$, whence $xy \in GV\langle y\rangle$ and

$xy \in GV\langle x\rangle \cup GV\langle y\rangle$. That $yx \in GV\langle x\rangle \cup GV\langle y\rangle$ can be proved dually.

By the above lemmas and lemma 1.2, we have the following lemma.

Corollary 2.5 Let S be a GV-semi group. If $SubGVS$ is 0-modular, then

$\kappa = \bigcup_{e\in Es}(K_e \times K_e)$ is a congruence on S if and only if S is a band of unipotent epi groups $K_e(e \in Es)$.

By the above lemmas, we can state and verify the main theorem of the section.

Theorem 2.6 Let S be a GV-semi group. Then $SubGVS$ is 0-modular if and only if S is a GU-band of unipotent epi groups $K_e(e \in Es)$.

Proof. The necessity can be easily proved by lemmas 2.1, 2.3 and 2.4.

To prove the sufficiency, let S be a GU-band of unipotentepi groups $K_e(e \in Es)$ and $A, B, C \in SubGVS$ such that $A \cap B = \phi$, $C \leq A$. We can conclude B and C lie in two different unipotentepi groups by $C \cap B = \phi$. Hence

$GV\langle B, C\rangle = B \cup C$, and so $A \cap (B \vee C) = A \cap (B \cup C) = C$, thatis, $SubGVS$ is 0-modular as required.

From the theorem, we can obtain the corresponding result on a completely regular semi group.

Theorem 2.7 Let S be a completely regular semi group, $SubCRS$ is0-modular if and only if S is a CU-band of the maximal subgroups G_e of S forall $e \in Es$.

3. The case of 0-distributivity

Theorem 3.1 Let S be a GV-semi group and $SubGVS$ 0-distributivity if and only

if $\{K_e, K_f\}$ is a left(right) zero band or chain for any $e, f \in Es$ with $e \neq f$.

To prove the theorem, we should use the following lemma.

Lemma 3.2 Let S be a GV-semi group and $\{K_e, K_f\}$ a left (right) zero band or chain, then $K_e \vee K_f = K_e \cup K_f$ for any $e, f \in Es$ with $e \neq f$.

Proof. For any $a \in K_e, b \in K_f$ with any $e, f \in Es, e \neq f$, then

$ab, ba \in K_e \cup K_f$ According to the fact that $\{K_e, K_f\}$ is a left (right) zero band or chain, hence

$\{K_e, K_f\} = K_e \cup K_f$, and so $K_e \vee K_f = \langle K_e, K_f\rangle = K_e \cup K_f$ for $K_e, K_f \in SubGVS$.

Now, we begin to prove theorem 3.1 using the above lemma.

Proof. To prove the necessity. We first put

$A = GV\langle ef\rangle$, $B = GV(e) = \{e\}$, $C = GV(f) = \{f\}$ for $e, f \in Es$ with $e \neq f$ and assume $A \cap B = A \cap C = \phi$, then $A \cap (B \vee C) = GV(ef) \cap GV(e, f) = \phi$ by 0-modularity of $SubGVS$. Thus $A \cap B = \phi$ or $A \cap C = \phi$.

If $A \cap B = \phi$, that is, $e \in GV\langle ef\rangle$. And there exists $g \in Es$ such that
$ef \in K_g$ since S is a GV-semi group. Without loss of generality, assume $g \neq e$.
Then $e \in GV\langle ef\rangle \subseteq K_g$ and $e \in K_e$, hence $e \in K_g \cap K_e$ with $g \neq e$, and so,
it leads to a contradiction. Thus $e = g, ef \in K_e$. Dually, we can get $ef \in K_f$ when $A \cap C = \phi$.
Suppose there exist $a \in K_e, b \in K_f$ such that $ab \notin K_e \cup K_f$ with
$e, f \in Es, e \neq f$. Then there exists $g \in Es \setminus \{e, f\}$ such that $ab \in K_g$ and

$(ab)^n \in K_g$ Since S is a GV-semi group. Hence $(ab)^n \notin K_e \cup K_f$ for any $n \in Z^+$, and so $GV\langle ab\rangle \cap GV\langle a\rangle = GV\langle ab\rangle \cap GV\langle b\rangle = \phi$. By 0-modularity of $SubGVS$, we get $GV\langle ab\rangle \cap GV\langle a, b\rangle = \phi$.

Obviously, it is a contradiction and so $K_eK_f \subseteq K_e \cup K_f$ .Symmetrically, we can prove $K_f K_e \subseteq K_e \cup K_f$ for any $e, f \in Es$, $e \neq f$.

Let $ef \in K_e$ for any $e, f \in Es$ with $e \neq f$ and assume there exist
$a \in K_e$, $b \in K_f$ such that $ab \in K_f$. Hence $af = (ab)b^{r(b)-1}(b^{r(b)})^{-1} \in K_f G_f \subseteq G_f$,
$a^n f = aa\cdots f^n = (af)^n$, $a^n = a^n e$ and $a^n ef = a^n f = (af)^n \in K_f$ for any $n \geq r(a)$,hence $K_e \cap K_f \neq \phi$ by $a^n ef = a^n(ef) \in K_e$. Thus $K_eK_f \subseteq K_e$ .By the same way, we have $K_eK_f \subseteq K_f$ if $ef \in K_f$ for any $e, f \in Es$ with $e \neq f$. On the other hand, K_e is a subsemi group of S for any $e \in Es$ since S is a GV-semi group, therefore $K_eK_e \subseteq K_e$. Consequently, we have $\{K_e, K_f\}$ is aleft (right) zero band or chain for any $e, f \in Es$ with $e \neq f$.

Next To prove the sufficiency. Let $A, B, C \in SubGVS$ with $A \cap B = A \cap C = \phi$, then $E_A \cap E_B = E_A \cap E_C \neq \phi$. For any $a \in A$,we have $e = a^{r(a)}(a^{r(a)})^{-1} \in A$,that is,
$e \in E_A$ and $A \in K_e$, thus $A \subseteq \cup_{e \in E_A} K_e$. Dually, $B \subseteq \cup_{e \in E_B} K_e$ and $C \subseteq \cup_{e \in E_C} K_e$.

By lemma 3.2, we get $B \vee C \subseteq (\bigcup_{e \in E_A} K_e) \cup (\bigcup_{e \in E_C} K_e)$ by $K_e \in SubGVS$ for any $e \in Es$ .Hence $A \cap (B \vee C) \subseteq \bigcup_{e \in E_A} K_e \cap [(\bigcup_{e \in E_B} K_e) \cup (\bigcup_{e \in E_C} K_e)] = \phi$ since $E_A \cap E_B = E_A \cap E_C = \phi$ and $e \neq f$ implies $K_e \cap K_f = \phi$. Thus $A \cap (B \vee C) = \phi$.

From the above theorem, we immediately obtain the next result on a completely regular semi group.

Theorem 3.3 For a completely regular semi group S , the lattice $SubCRS$ is 0-distributive if and only if $\{G_e, G_f\}$ is a left (right) zero band or chain for any $e, f \in Es$ with $e \neq f$.

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