

Biharmonic Anti-invariant Sub Manifolds in Kenmotsu Space Forms

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Abstract

We study in this paper the condition of biharmonicity of anti-invariant sub manifolds in Kenmotsu space forms.

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1. Introduction.

Harmonic maps on Riemannian manifolds have been studied for many years, starting with the paper of J. Eells and J.H. Sampson [9]. Due to their analytic and geometric properties, harmonic maps have become an important and attractive field of research.

The study of harmonic maps on Riemannian manifolds endowed with some structures has its origin in a paper of Lichnerowicz [16], in which he proved that a holomorphic map between Kähler manifolds is not only a harmonic map but also attains the minimum of energy in its homotopy class. After that, Rawnsley [18] studied structure preserving harmonic maps between f-manifolds. Later on Ianuș, Pastore, Gherghe, Chinea and some others (see [6], [12], [13], [14], [19]) studied harmonic maps defined on some almost contact manifolds (i.e. Sasakian, cosymplectic etc.).

The theory of biharmonic maps is an old and rich subject: they have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The Euler-Lagrange equation for bienergy functional was first derived by Jiange in 1986 [10]. After this biharmonic maps were studied by many authors see [2], [3], [4], [5], [8]. The purpose of this paper is to obtain a condition for biharmonicity of a map from anti-invariant sub manifolds of Kenmotsu manifolds to Kenmotsu space forms. After we recall some well-known facts about biharmonic maps and Kenmotsu manifolds, we prove the main results concerning biharmonic maps on anti-invariant sub manifolds of Kenmotsu manifolds.

2. Preliminaries

In this section, we recall some well-known facts concerning harmonic maps, biharmonic maps and Kenmotsu manifolds.

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Let $F: (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively. The energy density of F is a smooth function $e(F) : M \rightarrow [0, \infty)$ given by

$$e(F)_p = \frac{1}{2} \text{Tr}_g(F^*h)(p) = \frac{1}{2} \sum_{i=1}^m h(F_{*p}u_i, F_{*p}u_i),$$

For any $p \in M$ and any orthonormal basis $\{u_1, \dots, u_m\}$ of T_pM . If M is a compact Riemannian manifold, the energy $E(F)$ of F is the integral of its energy density:

$$E(F) = \int_M e(F) \nu_g,$$

Where ν_g is the volume measure associated with the metric g on M . A map $F \in C^\infty(M, N)$ is said to be harmonic if it is a critical point of the energy functional E on the set of all maps between (M, g) and (N, h) . Now, let (M, g) be a compact Riemannian manifold. If we look at the Euler-Lagrange equations for the corresponding variation problem, a map $F : M \rightarrow N$ is harmonic if and only if $\tau(F) \equiv 0$, where $\tau(F)$ is the tension field which is defined by

$$\tau(F) = \text{Tr}_g \nabla' dF$$

Where ∇' is the connection induced by the Levi-Civita connection on M and the F -pullback connection of the Levi Civita connection on N .

We take now a smooth variation $F_{s,t}$ with two parameters $s, t \in (-\epsilon, \epsilon)$ such that $F_{0,0} = F$. The corresponding variation vector fields are denoted by V and W . The second variation formula of E is:

$$H_F(V, W) = \frac{\partial^2}{\partial s \partial t} (E(F_{s,t})) |_{(s,t)=(0,0)} = \int_M h(J_F(V), W) \nu_g,$$

Where J_F is a second order self-adjoint elliptic operator acting on the space of variation vector fields along F (which can be identified with $\Gamma(F^{-1}(TN))$) and is defined by

$$J_F(V) = - \sum_{i=1}^m \left(\nabla'_{u_i} \nabla'_{u_i} - \nabla'_{\nabla'_{u_i} u_i} \right) V - \sum_{i=1}^m R^N(V, dF(u_i)) dF(u_i),$$

For any $V \in \Gamma(F^{-1}(TN))$ and any local orthonormal frame $\{u_1, \dots, u_m\}$ on M . Here R^N is the curvature tensor of (N, h) (see [11] for more details on harmonic maps).

J. Eells and L. Lemaire [11] proposed poly harmonic (k -harmonic) maps, and Jiang [10] studied the first and second variation formulas of biharmonic maps. Let us consider the bienergy functional defined by:

$$E_2(F_t) = \frac{1}{2} \int_M |\tau(F)|^2 \nu_g,$$

Where $|\nabla|^2 = h(\nabla, \nabla)$, $\nabla \in \Gamma(F^{-1}TN)$.

Then, the first variation formula of the bi energy functional is given by:

$$\frac{d}{dt} |_{t=0} E_2(F_t) = - \int_M h(\tau_2(F), V) \nu_g,$$

Here

$$\tau_2(F) = J(\tau(F)) = \Delta'(\tau(F)) - R(\tau(F)),$$

Which is called the bitension field of F and J is given by (1).

A smooth map F of (M, g) into (N, h) is said to be biharmonic if $\tau_2(F) = 0$.

Contact Manifolds are introduced detailed in [1]. However Tanno [20] has classified, into three classes, the connected almost contact Riemannian manifolds whose auto orphisms groups have the maximum dimensions:

- (1) Homogeneous normal contact Riemannian manifolds with constant φ - holomorphic sectional curvature;
- (2) Global Riemannian products of a line or a circle and a *Kahler* space form;
- (3) Warped product spaces $L \times_f N$, where L is a line and N a *Kahler* manifold.

Kenmotsu [15] studied the third class and characterized it by tensor equations. A (2m+1)-dimensional Riemannian manifold (M, g) is said to be a Kenmotsu manifold if it admits an endomorphism φ of its tangent bundle TM, a vector field ξ and a 1-form η ,

which satisfy

$$\begin{aligned} \varphi^2 &= -I + \eta \otimes \xi, & \eta(\xi) &= 1, & \varphi(\xi) &= 0, & \eta \circ \varphi &= 0, \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), & \eta(X) &= g(X, \xi), \\ (\Delta_X \varphi)Y &= -g(X, \varphi Y)\xi - \eta(Y)\varphi X, \end{aligned} \tag{5}$$

for any vector fields X, Y on M, where ∇ denotes the Riemannian connection with respect to g.

Example 2.1. Let N be a *Kahler* manifold, with the *Kahlerian* structure (J; h) and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(t) = ce^t$, where $c \in \mathbb{R}$, $c > 0$. Then the warped product $M = \mathbb{R} \times_f N$ is defined as being the manifold $\mathbb{R} \times N$ endowed with the Riemannian metric

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t)h_x \end{pmatrix}.$$

If we put $\xi = \frac{d}{dt}$, $\eta(X) = g(X, \xi)$ and

$$\varphi(t, x) = \begin{pmatrix} 0 & 0 \\ 0 & \exp((t\xi))_* J_{(x)} \exp((-t\xi))_* \end{pmatrix},$$

for any point $(t, x) \in \mathbb{R} \times N$ and any vector field X tangent to M, then M is Kenmotsu manifold [15].

Definition 2.1. [17] : A submanifold M of a Kenmotsu manifold N is a normal semi-invariant submanifold if ξ is normal to M and M has two distributions D and D^\perp , called the invariant, respectively, the anti-invariant distribution of M so that

1. $T_x M = D_x \oplus D^\perp_x$;
2. $D_x, D^\perp_x, \langle \xi_x \rangle$ are orthogonal;
3. $\varphi D_x \subseteq D_x$; $\varphi D^\perp_x \subseteq D^\perp_x$, for all $x \in M$.

If $D = 0$, then M is a normal anti-invariant submanifold of N and if $D^\perp = 0$, then M is an invariant submanifold of N .

3. Main Results

Before the main results, recall the following results by Jiang:

Lemma 3.1. [10] Let $f: (M^m, g) \rightarrow (N^n, h)$ be an isometric immersion whose mean curvature vector field $H = \frac{1}{m} \tau(f)$ is parallel; $\nabla^\perp H = 0$, where ∇^\perp is the induced connection of the normal bundle $T^\perp M$ by f . Then

$$\bar{\Delta} \tau(f) = \sum_{i=1}^m h(\bar{\Delta} \tau(f), df(e_i)) df(e_i) - \sum_{i,j=1}^m h(\bar{\nabla}_{e_i} \tau(f), df(e_j)) (\bar{\nabla}_{e_i} df)(e_j),$$

where $\{e_i\}$ is a locally defined orthonormal frame field of (M, g) .

Lemma 3.2. [10] Let $f: (M^m, g) \rightarrow (N^n, h)$ be an isometric immersion whose mean curvature vector field $H = \frac{1}{m} \tau(f)$ is parallel; $\nabla^\perp H = 0$, where ∇^\perp is the induced connection of the normal bundle $T^\perp M$ by f . Then,

$$\tau(f) = - \sum_{j,k=1}^m h(\tau(f), R^N(df(e_j), df(e_k))) df(e_j) df(e_k) - \sum_{i,j=1}^m h(\tau(f), (\bar{\nabla}_{e_i} df)(e_j)) (\bar{\nabla}_{e_i} df)(e_j),$$

where $\{e_i\}$ is a locally defined orthonormal frame field of (M, g) .

Lemma 3.3. [10] Let $f: (M^m, g) \rightarrow (N^{m+1}, h)$ be an isometric immersion which is not harmonic. Then, the condition that $\|\tau(f)\|$ is constant is equivalent to the one that

$$\bar{\nabla}_X(\tau(f)) \in \Gamma(f_* TM), \quad \text{for all } X \in TM,$$

that is the mean curvature tensor is parallel with respect to ∇^\perp . For details and proof of these Lemmas, see [8], [10]. Now the main result of this article;

Theorem 3.1. Let (M, g) be a real n -dimensional anti-invariant submanifold of Kenmotsu space form N of dimension $(2n+1)$, and $F: (M, g) \rightarrow (N, h)$ be an isometric immersion with non zero constant parallel mean curvature with respect to connection on normal bundle, then necessary and sufficient conditions for F to be biharmonic is $\tau \in \phi TM$ and

$$\|B(F)\|^2 = \left(\frac{n(c-3)}{4} + 3 \frac{(c-1)}{4} \right), \quad c > \frac{3(n+1)}{n+3}.$$

Proof. A Kenmotsu manifold with constant ϕ -sectional curvature c is called a Kenmotsu space form and its curvature tensor R is expressed by

$$R(X, Y)Z = \frac{c-3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4} \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}.$$

By Lemma 3.1, the mean curvature vector field of F is parallel with respect to ∇^\perp , therefore we apply Lemma 3.1 and Lemma 3.2. Let $\{e_i\}_{i=1}^n$ be orthonormal basis on M , we have

$$R^N(dF(e_j), dF(e_k)) di(e_k) = \frac{c-3}{4} \{ndF(e_j) - \delta_{jk}dF(e_k)\} + \frac{c-1}{4} 3dF(e_j).$$

Then we have

$$\sum_{j,k=1}^m h(\tau(F), R^N(dF(e_j), dF(e_k)) dF(e_k) dF(e_j) = 0.$$

And

$$\bar{\Delta} \tau(F) = \sum_{i,j=1}^m h(\tau(F), (\tilde{\nabla}_{e_i} dF) e_j) (\tilde{\nabla}_{e_i} dF) e_j$$

Furthermore, we have

$$R(\tau(F)) = \left\{ n \frac{c-3}{4} + 3 \frac{c-1}{4} \right\} \tau(F) - \frac{c-1}{4} (n+3) \eta(\tau(F)) \xi. \tag{6}$$

For anti-invariant submanifolds, we consider the decomposition $T_x^\perp M = \varphi T_x M \oplus \langle \xi_x \rangle$,

Since $TN = TM \oplus TM^\perp = TM \oplus \varphi TM \oplus \xi$, now if $\tau(F) \in \varphi TM$, then

$$R(\tau(F)) = \left\{ n \frac{c-3}{4} + 3 \frac{c-1}{4} \right\} \tau(i).$$

Now if $\tau(F) \in \langle \xi \rangle$, then $\tau(F) = a\xi$, where a is any constant and

$$R(\tau(F)) = \left\{ n \frac{c-3}{4} + 3 \frac{c-1}{4} \right\} a\xi - \frac{c-1}{4} (n+3) a\xi = -\frac{n}{2} \xi,$$

If $\tau(F) \in \varphi TM \oplus \langle \xi \rangle$, then

$$R(\tau(F)) = \left\{ n \frac{c-3}{4} + 3 \frac{c-1}{4} \right\} \tau(F) - \frac{c-1}{4} (n+3) \eta(\tau(F)) \xi.$$

Now the necessary and sufficient conditions F to be biharmonic is that

$$\tau_2(F) = \bar{\Delta} \tau(F) - R(\tau(F)) = 0.$$

This becomes

$$\begin{aligned} \sum_{j,k=1}^m h(\tau(F), \tilde{\nabla}_{e_j} dF(e_k)) (\tilde{\nabla}_{e_i} dF(e_k) - \left\{ n \frac{c-3}{4} + 3 \frac{c-1}{4} \right\} \tau(F) + \frac{c-1}{4} (n+3) \eta(\tau(F)) \xi) \\ = 0. \end{aligned} \tag{7}$$

Now let

$$B(F)(e_j, e_k) = (\tilde{\nabla}_{e_j} dF)(e_k) = h(dF(e_j), dF(e_k))V = h_{jk}V,$$

where V is the unit normal vector along $F(M)$. Then

$$\tau(F) = \sum_{r=1}^n (\tilde{\nabla}_{e_r} dF)(e_r) = \sum_{r=1}^n h_{rr}V,$$

where V is the unit normal vector along $F(M)$. Thus, the left hand side of (7) becomes as:

$$\begin{aligned} \sum_{j,k,r=1}^m \left\{ h_{rr}h_{jk}h_{jk}V - \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right) h_{rr}V + \frac{c-1}{4}(n+3)h_{rr}h(V, \xi)\xi \right\} &= 0, \\ \left(\sum_{r=1}^m h_{rr} \right) \sum_{j,k=1}^m \left\{ h_{jk}h_{jk}V - \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right) V + \frac{c-1}{4}(n+3)h(V, \xi)\xi \right\} &= 0, \\ \|\tau(F)\| \left\{ \|B(F)\|^2 V - \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right) V + \frac{c-1}{4}(n+3)h(V, \xi)\xi \right\} &= 0. \end{aligned}$$

Now if $\tau(F) \in \phi TM$, then $V \perp \xi$ and we have

$$\|B(F)\|^2 - \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right) = 0,$$

and

$$\|B(F)\|^2 = \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right). \quad (8)$$

Now If $\tau(F) \in \langle \xi \rangle$, then $V \perp \phi TM$ and $V = a\xi$, a is any real constant, then we have

$$\|\tau(F)\| \left\{ \|B(F)\|^2 a\xi - \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right) a\xi + \frac{c-1}{4}(n+3)h(V, \xi)a\xi \right\} = 0,$$

and

$$\|B(F)\|^2 = -\frac{n}{2}. \quad (9)$$

From (9), it is clear that τ does not belong to $\langle \xi \rangle$ and $\tau \in \phi TM$, for biharmonic anti-invariant immersion and satisfies

$$\|B(F)\|^2 = \left(n\frac{c-3}{4} + 3\frac{c-1}{4} \right), \quad (10)$$

which implies $c > \frac{3(n+1)}{n+3}$.

Remark 3.1: One of the normal anti-invariant n -dimensional submanifold M in $(2n+1)$ -dimension Kenmotsu manifold N is Whitney sphere [7]. Thus a map from Whitney sphere in Kenmotsu manifold to Kenmotsu manifold of constant ϕ -sectional curvature c is biharmonic if its second fundamental form satisfy relation (10).

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