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On the Discriminates of map germs from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n}$, n = 3; 4.

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Abstract

In this work we show some calculations involving Lê numbers and Euler characteristic of the Milnor fibre on discriminates of map germs from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n}$, n = 3; 4. In particular, we show that the Lê numbers are not invariant to the class of weight homogeneous map germs with same degree of homogeneity.

Keywords and Phrases: Lê numbers, Euler characteristic, Milnor fibre.

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2. Lê numbers

To define and describe the Lê numbers we follow the notation and approach made by Massey in [4]. We assume that the reader is familiar with the notion of coherent sheaves, gap sheaves, schemes, and cycles. In order to fix the notation, for a sheaf α and an analytic subset W in some affine space, we denote by α/W the corresponding gap sheaf and by $V(\alpha)/W$ the scheme associated with the sheaf α/W, where V(α) denotes the analytic space defined by the vanishing of α. We shall at times enclose cycles in square brackets, [·].

For the definitions in this section let h: (U; 0) → ($\mathbb{C}$; 0) be an analytic function, U an open subset of $\mathbb{C}^{n+1}$ containing the origin and z = (z_0; z_1; …; z_N) a linear choice of coordinates in $\mathbb{C}^{N+1}$. We fix the dimension of the singular set Σ(h) of h is s with $0 \leq s \leq N$.

Definition 2.1: For $0 \leq k \leq N$, the k^{th} (relative) polar variety, $\Gamma^k_{h,z}$, of h with respect to z is the scheme V($\frac{\partial h}{\partial z_k}$, ..., $\frac{\partial h}{\partial z_N}$) /Σ(h).

On the level of ideals, $\Gamma^k_{h,z}$ consists of the components of V ($\frac{\partial h}{\partial z_k}$, ..., $\frac{\partial h}{\partial z_N}$) which are not contained in Σ(h). We denote by [$\Gamma^k_{h,z}$] the cycle associated with this scheme. In particular we note that $\Gamma^0_{h,z}$ is empty and we call $\Gamma^{N+1}_{h,z}$ = U.

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Definition 2.2: For 0 $\leq$ k $\leq$ N, the k^{th} Lê cycle $\Lambda^k_{h,z}$, of h with respect to z is the cycle

$$\left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right] - \left[\Gamma^k_{h,z}\right].$$

In general we shall denote this cycle simply $\Lambda^k_{h,z}$, and not [$\Lambda^k_{h,z}$], because unlike the polar varieties which are defined as schemes and we have to consider the associated cycle, this definition is given in terms of cycles.

If the intersection of $\Lambda^k_{h,z}$ with the cycle of V ($z_0 - p_0$; . . . ; $z_{k-1} - p_{k-1}$) is purely 0-dimensional at a point p = (p_0; p_1; … ; p_N), i.e., either p is an isolated point of the intersection or p is not in the intersection, it is possible to define the Lê numbers as follows:

Definition 2.3: For 0 $\leq$ k $\leq$ N, the k^{th} Lê number, $\lambda^k_{h,z}$(p), of h with respect to z at p, is defined as the intersection number

$\Lambda_{h,z}.V(z_0 - p_0;\ldots;z_{k-1} - p_{k-1})_p$

Remark 2.4: It is possible to give a more intuitive characterization of $\lambda^s_{h,z}$(p), s = $dim_p\Sigma$ h. Assuming that $\lambda^s_{h,z}$(p) exists, by moving to a generic point, we can be to show that $\lambda^s_{h,z}$(p)$= \sum_\nu n_\nu \mu^0_\nu$, where ν runs over all s-dimensional components of Σ h in p, n_ν is the local degree of the map (z_0; z_1; ... ; z_s) restricted to ν at p, and μ^0_ν denotes the generic transverse Milnor number of h along the component ν in a neighborhood of p. In particular, if the coordinate system is generic enough so that n_ν is actually the multiplicity of ν at p for all ν, then $\lambda^s_{h,z}$(p) is merely the **multiplicity of the Jacobean scheme** of h (the scheme defined by the vanishing of the Jacobian ideal) at p.

Theorem 2.5 ([4], Theorem 1.28). Let p $\in$ V (h) and let s = $dim_p\Sigma$ h. Then, for a generic choice of coordinates, all of the Lê number and polar numbers of h at p in dimensions less than or equal to s are defined.

Theorem 2.6 ([4], Theorem 3.3). Let U be an open subset of $\mathbb{C}^{n+1}$, h : $\mathcal{U} \to \mathbb{C}$ be an analytic map, p in V (h), s :=$dim_p\Sigma$ h and z = ($z_0, \ldots, z_{s-1}$) be prepolar for h at p.

1.If s $\leq$ n $-$ 2, then $F_{h,p}$ is obtained up to diffeomorphism from a real 2n- ball by successively attaching $\lambda^{n-k}_{h,z}$ (p) k-handles, where n $-$ s $\leq$ k $\leq$ n.

2. If s = n $-$ 1, then $F_{h,p}$ is obtained up to diffeomorphism from a real 2n- manifold with the homotopy -type of a bouquet of $\lambda^{n-1}_{h,z}$ (p) circles by successively attaching $\lambda^{n-k}_{h,z}$ (p) k-handles, where 2 $\leq$ k $\leq$ n.

Hence, the reduced Euler characteristic of the Milnor fibre of h at p is given by

$$\tilde{X}(F_{\mathrm{h,p}}) = \sum_{i=0}^{s} (-1)^{n-i}\, \lambda^{i}_{\mathrm{h,z}}(\mathrm{p}).$$

Corollary 2.7: Let h be a homogeneous polynomial of degree d in n + 1 variables, let s: $= dim_0 \Sigma$ h, and suppose that $\lambda^{i}_{\mathrm{h,z}}(0)$ exists for all i $\leq$ s. Then,

$$\sum_{i=0}^{s} (\mathrm{d}-1)^{i}\, \lambda^{i}_{\mathrm{h,z}} = (\mathrm{d}-1)^{n+1}.$$

3. Euler characteristic of the Milnor fiber of weight-homogeneous

Let f: $\mathbb{C}^{n+1} \to \mathbb{C}$ be a weight homogeneous polynomial of type (r_0, r_1, ..., r_n; d).

If π: $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is given by $\pi\ (z_0, z_1, \ldots,\ z_n) = (z_0^{r_0}, z_1^{r_1}, \ldots, z_n^{r_n}),$

Then, h := $f\ o\ \pi$ is homogeneous.

Moreover, the restriction of π induces a map, $\tilde{\pi}$, from the Milnor fibre, $F_{\mathrm{h,0}}$, of h at the origin to the Milnor fibre , $F_{\mathrm{f,0}}$ of f at the origin; , $\tilde{\pi}$: $F_{\mathrm{h,0}} \to F_{\mathrm{f,0}}$.

Now, let I denote the indexing set $\{0, \ldots, \mathrm{n}\}$ and for each J $\subseteq$ I, let w_J denote the intersection of hyper planes given by $w_J := \mathrm{V}\,(\mathrm{z}_j : \mathrm{j} \in \mathrm{J})$; and let S_J denote the Whitney stratum

$$S_J := w_J - \bigcup_{J \subsetneq K} w_K.$$

Note that $\{S_J\}$ determines Whitney stratifications $\{S_J \cap F_{\mathrm{h,0}}\}$ and $\{S_J \cap F_{f,0}\}$ de $F_{\mathrm{h,0}}$ and $F_{f,0}$, respectively, and with these stratifications, $\tilde{\pi}$ becomes a stratified map. Moreover, the restriction of $\tilde{\pi}$ to a map from $S_J \cap F_{\mathrm{h,0}}$ to $S_J \cap F_{\mathrm{f,0}}$ is a topological covering map with fibre equal to $\prod_{i \notin J} r_i$ points.

Hence,

$$\chi(F_{\mathrm{h,o}}) = \sum_{J} \chi(S_J \cap F_{\mathrm{h,0}}) = \sum_{J} \left(\prod_{i \notin J} r_i \right) \chi(S_J \cap F_{\mathrm{f,0}}).$$

Some elementary combinatorial shows that this last quantity is equal to

$$\left(\sum_{J} c_J \sum_{J \subseteq K} \chi(S_K \cap F_{\mathrm{f,0}}) \right).$$

Where

$$c_J \;=\; (\;\;1)^{|I|}\sum_{L\subseteq J}\left((\;\;1)^{|I|}\prod_{i\notin L} r_i\right).$$

The advantage of this last form is that $\displaystyle\sum_{J\subseteq K}\chi(S_K \cap F_{f,0}) = \chi\left(F_{f_{|w_J}},\; 0\right).$

Therefore, we have that

$$\chi(F_{h,0}) = \sum_J c_J\,\chi\left(F_{f_{|w_J}},\; 0\right)$$

where c_J is as above.

It follows that $\displaystyle\chi(F_{h,0}) = (r_0 \dots\dots\dots r_n)\;\chi(F_{f,0}) \;+\; \sum_{J\neq\emptyset} c_J\;\;\chi\left(F_{f_{|w_J}},\; 0\right),$

and finally, we arrive at the formula

$$\chi(F_{f,0}) = \frac{\chi(F_{h,0}) \;-\;\; \Sigma_{J\neq\emptyset}\, c_J\;\;\chi\left(F_{f_{|w_J}},\; 0\right)}{r_0 \dots\dots\dots r_n} \qquad (I)$$

Example 3.1: Euler characteristic of the Milnor fiber of the weight-homogeneous plane curve

Suppose that the irreducible factorization of f(z_0; z_1) is $z_0^a z_1^b \prod f_i^{m_i}$, where we allow for the case where a or b equals a 0. Let π(z_0, z_1) = ($z_0^{r_0}$,$z_1^{r_1}$), and let h denote the homogeneous polynomial $f \circ \pi$. Let h_i denote the homogeneous polynomial $f_i \circ \pi$. Let d be the degree h and let d_i be the degree of h_i. Then, the formula (I) becomes

$$\chi(F_{f,0}) = \;=\frac{\chi(F_{h,0}) \;+\; (r_0 \;\;-1)\,r_1\;\chi\left(F_{f_{|V(z_0)}},\; 0\right) \;+\; (r_1\;\;-1)\,r_0\;\chi\left(F_{f_{|V(z_1)}},\; 0\right)}{r_0 r_1}$$

Now, $\chi\left(F_{f_{|V(z_k)}},\; 0\right) = \;0$ if $f_{|V(z_k)} \equiv 0$ and simply equals the multiplicity of $f_{|V(z_k)}$ otherwise. In addition, as h is homogeneous, we may calculate $\chi(F_{h,0})$ by knowing only $\lambda_h^1\,(0)$.

Consequently,

$$r_0 r_1\, \chi(F_{f,0}) = \begin{cases} -\mathrm{d} \sum \mathrm{d}_i & \text{if} \quad a \neq 0,\ b \neq 0 \\ \mathrm{d}\left(r_0 - \sum \mathrm{d}_i\right), & \text{if} \quad a = 0,\ b \neq 0 \\ \mathrm{d}\left(r_1 - \sum \mathrm{d}_i\right), & \text{if} \quad a \neq 0,\ b = 0 \end{cases}$$

and
$$r_0 r_1\, \chi(F_{f,0}) = \mathrm{d}\left(r_0 + r_1 - \sum \mathrm{d}_i\right), \qquad \text{if} \quad a = 0,\ b = 0.$$

4. Euler characteristic of the Milnor fibre on discriminates of map germs

In this section we will use the section 3 to calculate the Euler characteristic of the Milnor fibre on discriminates of some map germs from $\mathbb{C}^4$ to $\mathbb{C}^3$.

4.1 Euler characteristic of the Milnor fibre of the $f = (x; y; u; xu + yu + u^3 + v^3)$.

Let weight homogeneous map germ $f: (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ defined for f(x; y; u; v) = $(x; y; u; xu + yu + u^3 + v^3)$.

From **Aldicio's algorithm** [1] follows that the discriminant this map germ is given as the locus zero of the map germ $g: (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ defined by

$$g(X_1, X_2, X_3) = 729\, X_3^4 + 216 X_1^3\, X_3^2 + 216 X_2^3\, X_3^2 + 16 X_1^6 - 32 X_1^3\, X_2^3 + 16\, X_2^6\,.$$

This map germ is weighted homogeneous polynomial whose variables have weights equal r_0 = 2; r_1 = 2; r_2 = 3 respectively and its degree is $\mathrm{d} = 12$.

Using the notation of section 3 and the example 3.1, we find $c_0 = ((-1))(r_0 r_1 r_2 - r_1 r_2) = -6$,

$$f_{|W_{\{0\}}} = (27 X_3^2 + 4 X_2^3)^2.$$

Hence, $\chi\left(F_{f_{|W_{\{0\}}}},\ 0\right) = (12(2+3-6))/6 = -2$.

Now $c_1 = ((-1))(r_0 r_1 r_2 - r_0 r_2) = -6$ and $f_{|W_1} = (27 X_3^2 + 4 X_1^3)^2$.

So, $\chi\left(F_{f_{|W_{\{1\}}}},\ 0\right) = (12(2+3-6))/6 = -2.$

We have $c_2 = ((-1))(r_0 r_1 r_2 - r_0 r_1) = -6$ and $f_{|W_2} = 16(X_1^3 - 4 X_2^3)^2$.

Then, $\chi\left(F_{f_{|W\{\,2\}}},\ 0\right) = \left(12(2+3-6)\right)/4 = -6$.

Now, [illegible] and [illegible] .

Hence, $\chi\left(F_{f_{|W\{0,1\}}},\ 0\right) = 4.$.

We calculate, [illegible] and $f_{|W\{0,2\}} = 16X_2^6$.

So, $\chi\left(F_{f_{|W\{0,2\}}},\ 0\right) = 6.$.

We have, [illegible] and $f_{|W\{1,2\}} = 16X_1^6$.

Then, $\chi\left(F_{f_{|W\{1,2\}}},\ 0\right) = 6.$.

If we define $\pi(X_1, X_2, X_3) = (X_1^2, X_2^2, X_3^3)$, then

$$h(X_1,X_2,X_3) = (f\circ\pi)(X_1,X_2,X_3) = [illegible]$$

is a homogeneous polynomial of degree 12. To get the Euler characteristic of the Milnor fibre of the f at the origin, $\chi(F_{f,0})$, we need calculate $\chi(F_{h,0})$.

Now, we observe dim Σh = 1, from the theorem 2.5, the Lê numbers $\lambda_h^0(0)$ and $\lambda_h^1(0)$ exist. Moreover, using software SINGULAR, we obtained that the multiplicity of the Jacobian scheme of h at the origin is 90; so by remark 2.4, its follows $\lambda_h^1(0)$ = 90. Since d = 12 and $\lambda_h^1(0)$ = 90, its follows from corollary 2.7 that $\lambda_h^0(0) = 341$. Consequently, by applying the theorem 2.6, we get

$$\chi(F_{h,0}) = \lambda_h^0(0) - \lambda_h^1(0) + 1 = 341 - 90 + 1 = 252$$

About these facts above we finally obtained,

$$\chi(F_{f,0}) = \frac{252 - ((-6)(-2) + (-6)(-2) + (-8)(-6) + 3.4 + 4.6 + 4.6)}{12}$$
$$= (252 - 132)/12 = 120/12 = 10.$$

4.2 Euler characteristic of the Milnor fibre of the $f = (x; y; u; xu + yu + u^3 + uv^2)$.

Let weight homogeneous map germ f:$(\mathbb{C}^4, 0) \longrightarrow (\mathbb{C}^3, 0)$ defined for

$$f(x;\ y;\ u;\ v) = (x; y; u; xu + yu + u^3 + uv^2).$$

From **Alnico's algorithm** [1] follows that the discriminate this map germ is given as the locus zero of the map germ $g: (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ defined by

$$g(X_1, X_2, X_3) = 27\,X_3^4 + 36\,X_1^2\,X_2 X_3^2 + 4X_2^3 X_3^2 + 4X_1^6 + 8X_1^4\,X_2^2 + 4\,X_1^2 X_2^4.$$

This map germ is weighted homogeneous polynomial whose variables have weights $r_0 = 2$; $r_1 = 2$; $r_2 = 3$ respectively and its degree is d = 12.

Using the notation of section 3 and the example 3.1, we find [illegible] and

$$f|_{W\{0\}} = X_3^2\ (27X_3^2 + 4X_2^3).$$

Hence, $\chi\left(F_{f|_{W\{0\}}},\ 0\right) = (12(2-6))/6 = -8.$

Now, [illegible] and $f|_{W\{1\}} = (27X_3^4 + 4X_1^6).$

Then, $\chi\left(f|_{W\{1\}},\ 0\right) = (12(2+3-12))/6 = -14.$

We calculate, [illegible] and $f|_{W\{2\}} = 4X_1^2\ (X_1^2 + X_2^2)^2.$

So, $\chi\left(f|_{W\{2\}},\ 0\right) = (12(2-4))/4 = -6.$

We have, [illegible] and $f|_{W\{0,1\}} = 27X_3^4.$

Hence, $\chi\left(F_{f|_{W\{0,1\}}},\ 0\right) = 4.$

We obtain [illegible] and $f|_{W\{0,2\}} \equiv 0.$

Then, $\chi\left(F_{f|_{W\{0,2\}}},\ 0\right) = 0.$

Now, [illegible] and $f|_{W\{1,2\}} = 4X_1^6.$

Therefore, $\chi\left(F_{f|_{W\{1,2\}}},\ 0\right) = 6.$

Then, if we define $\pi(X_1, X_2, X_3) = (X_1^2, X_2^2, X_3^3)$, consequently

is a homogeneous polynomial of degree 12. To get the Euler characteristic of the Milnor fibre of the f at the origin, $\chi(F_{f,0})$, we need calculate $\chi(F_{h,0})$.

Now, we observe dim Σh = 1, from the theorem 2.5, the Lê numbers $\lambda_h^0(0)$ and $\lambda_h^1(0)$ exist. Moreover, using software SINGULAR, we obtained that the multiplicity of the Jacobian scheme of h at the origin is 83; so by remark 2.4, its follows $\lambda_h^1(0)$ = 83. Since d = 12 and $\lambda_h^1(0)$= 83, its follows from corollary 2.7 that $\lambda_h^0(0)$ is equal 418.

Consequently, by applying the theorem 2.6, we get $\chi(F_{h,0}) = \lambda_h^0(0) - \lambda_h^1(0) + 1 = 418 - 83 + 1 = 336.$

About these facts above we finally obtained,

$$\chi(F_{f,0}) = \frac{336 - ((-6)(-8) + (-6)(-14) + (-8)(-6) + 3.4 + 4.0 + 4.6)}{12}$$
$$= (336 - 216)/12 = 120/12 = 10.$$

Remark 4.1: Note that for the map germ of the section 4.1 we have $\lambda_h^1(0)$ = 90 while for the map germ of the section 4.2 we have $\lambda_h^1(0)$ = 83, with this we can conclude that the Lê numbers are not invariant to the class of weight homogeneous map germ with same degree of homogeneity.

4.3 Euler characteristic of the Milnor fibre of the umbilic point, $\mathbf{D_4}$

Let weight homogeneous map germ f: $(\mathbb{C}^5, 0) \longrightarrow (\mathbb{C}^4, 0)$ defined by

From **Aldicio's algorithm** [1] follows that the discriminate this map germ is given as the locus zero of the map germ $g: (\mathbb{C}^4, 0) \longrightarrow (\mathbb{C}, 0)$ defined by

.

This map germ is weighted homogeneous polynomial whose variables have weights r_0 = 1; r_1 = 2; r_2 =2; r_3 = 3 respectively and its degree is d = 12.

Using the notation of section 3 and the example 3.1, we find

$c_{[illegible]} = (-1)(r_0 r_1 r_2 r_3 - r_1 r_2 r_3) = 0$ and $c_{[illegible]} = (-1)(r_0 r_1 r_2 r_3 - r_0 r_2 r_3) = -[illegible]$

$F_{f_{[illegible]}} = 16 X_4^3 (27 X_4^2 + 4 X_3^3 + 18 X_1 X_3 X_4 - X_1^2 X_3^2 - 4 X_1^3 X_4)$. Hence, $\chi\left(F_{f_{|w\{1\}}}, 0\right) = 10$.

Also, $c_{[illegible]} = (-1)(r_0 r_1 r_2 r_3 - r_0 r_1 r_3) = -[illegible]$ and
$F_{f_{[illegible]}} = 432 X_4^4 + 64 X_2^6 - 192 X_1 X_2^4 X_4 - 24 X_1^2 X_2^2 X_4^2 - 64 X_1^3 X_4^3 + 27 X_1^4 X_2^4$.

We obtain, $\chi\left(F_{f_{|w\{2\}}}, 0\right) = 34$.

Now, $c_{[illegible]} = (-1)(r_0 r_1 r_2 r_3 - r_0 r_1 r_2) = -[illegible]$ and

$F_{f_{[illegible]}} = 432 X_4^4 + 64 X_2^6 - 192 X_1 X_2^4 X_4 - 24 X_1^2 X_2^2 X_4^2 - 64 X_1^3 X_4^3 + 27 X_1^4 X_2^4$.

Therefore, $\chi\left(F_{f_{|w\{3\}}}, 0\right) = 36$.

Also, we calculate $c_{\{0,1\}} = (r_0 r_1 r_2 r_3 - r_1 r_2 r_3 - r_0 r_2 r_3 + r_2 r_3) = 0$

$c_{\{0,2\}} = (r_0 r_1 r_2 r_3 - r_1 r_2 r_3 - r_0 r_1 r_3 + r_1 r_3) = 0$
$c_{\{0,3\}} = (r_0 r_1 r_2 r_3 - r_1 r_2 r_3 - r_0 r_1 r_2 + r_1 r_2) = 0$

$c_{\{1,2\}} = (r_0 r_1 r_2 r_3 - r_0 r_2 r_3 - r_0 r_1 r_3 + r_0 r_3) = 3.$

$F_{f_{|w\{1,2\}}} = 16 X_4^3 (27 X_4 - 4 X_1^3)$. Hence, $\chi\left(F_{f_{|w\{1,2\}}}, 0\right) = -8$.

We have, $c_{\{1,3\}} = (r_0 r_1 r_2 r_3 - r_0 r_2 r_3 - r_0 r_1 r_2 + r_0 r_2) = 4$ and

$f_{|w\{1,3\}} \equiv 0$. Therefore, $\chi\left(F_{f_{|w\{1,3\}}}, 0\right) = 0$.

Now, $c_{\{2,3\}} = (r_0 r_1 r_2 r_3 - r_0 r_1 r_3 - r_0 r_1 r_2 + r_0 r_1) = 4$, and

$f_{|w\{2,3\}} = X_2^4 (64 X_2^2 + 27 X_1^4)$. Hence, $\chi\left(F_{f_{|w\{2,3\}}}, 0\right) = -18$.

We obtain,

$$c_{\{0,1,2\}} = -\left(r_0 r_1 r_2 r_3 - r_1 r_2 r_3 - r_0 r_2 r_3 - r_0 r_1 r_3 + r_2 r_3 + r_1 r_3 + r_0 r_3 - r_3\right) = 0$$

$$c_{\{0,1,3\}} = -\left(r_0 r_1 r_2 r_3 - r_1 r_2 r_3 - r_0 r_2 r_3 - r_0 r_1 r_2 + r_2 r_3 + r_1 r_2 + r_0 r_2 - r_2\right) = 0$$

$$c_{\{0,2,3\}} = -\left(r_0 r_1 r_2 r_3 - r_1 r_2 r_3 - r_0 r_1 r_3 - r_0 r_1 r_2 + r_1 r_3 + r_1 r_2 + r_0 r_1 - r_1\right) = 0$$

$$c_{\{1,2,3\}} = -\left(r_0 r_1 r_2 r_3 - r_0 r_2 r_3 - r_0 r_1 r_3 - r_0 r_1 r_2 + r_0 r_3 + r_0 r_2 + r_0 r_1 - r_0\right) = -2.$$

$f|_{W\{1,2,3\}} \equiv 0.$ Therefore, $\chi\left(F_{f|_{W\{1,2,3\}}},\ 0\right) = 0.$

Now if we define $\pi(X_1, X_2, X_3, X_4) = (X_1^2, X_2^2, X_3^2, X_4^3)$, then

$$\begin{aligned} \mathrm{h}(X_1, X_2, X_3, X_4) := (g \circ \pi)(X_1, X_2, X_3) \\ = 27X_1^4X_2^8 + 64X_2^{12} - 144X_1^2X_2^8X_3^2 + 128X_2^8X_3^4 - 16X_1^2X_2^4X_3^6 \\ + 64X_2^4X_3^8 \\ -192X_1X_2^8X_4^3 - 72X_1^3X_2^4X_3^2X_4^3 + 320X_1X_2^4X_3^4X_4^3 - 24X_1^2X_2^4X_4^6 \\ + 576X_2^4X_3^2X_4^6 - 16X_1^2X_3^4X_4^6 + 64X_3^6X_4^6 - 64X_1^3X_4^9 \\ + 288X_1X_3^2X_4^9 + 432X_4^{12}. \end{aligned}$$

Is a homogeneous polynomial of degree 12. To get the Euler characteristic of the Milnor fibre of the f at the origin, $\chi(F_{f,0})$, we need calculate $\chi(F_{\mathrm{h},0})$.

Now, we observe dim Σh = 2, from the theorem 2.5, the Lê numbers $\lambda_{\mathrm{h}}^0(0)$, $\lambda_{\mathrm{h}}^1(0)$ and $\lambda_{\mathrm{h}}^2(0)$ exist. Moreover, using software SINGULAR, we obtained that the multiplicity of the Jacobian scheme of h at the origin is 83; so by remark 2.4, its follows $\lambda_{\mathrm{h}}^2(0)$= 75. Since d = 12 and $\lambda_{\mathrm{h}}^2(0)$= 75, its follows from corollary 2.7 and from definition of the Lê number that $\lambda_{\mathrm{h}}^0(0)$ = 847 e $\lambda_{\mathrm{h}}^1(0)$= 429. Consequently, by applying the Theorem 2.6, we get $\chi(F_{\mathrm{h},0}) = -\lambda_{\mathrm{h}}^0(0) + \lambda_{\mathrm{h}}^1(0) - \lambda_{\mathrm{h}}^2(0) + 1 = -847 + 429 - 75 + 1 = -493.$

About these facts above we finally obtained,

$$\chi(F_{f,0}) = \left[\chi(F_{h,0}) - \left(\begin{array}{c} c_{\{0\}}\chi\left(F_{f|_{W\{0\}}},0\right) + c_{\{1\}}\chi\left(F_{f|_{W\{1\}}},0\right) + c_{\{2\}}\chi\left(F_{f|_{W\{2\}}},0\right) + c_{\{3\}}\chi\left(F_{f|_{W\{3\}}},0\right) + c_{\{0,1\}}\chi\left(F_{f|_{W\{0,1\}}},0\right) + \\ c_{\{0,2\}}\chi\left(F_{f|_{W\{0,2\}}},0\right) + c_{\{0,3\}}\chi\left(F_{f|_{W\{0,3\}}},0\right) + c_{\{1,2\}}\chi\left(F_{f|_{W\{1,2\}}},0\right) + c_{\{1,3\}}\chi\left(F_{f|_{W\{1,3\}}},0\right) + \\ c_{\{2,3\}}\chi\left(F_{f|_{W\{2,3\}}},0\right) + c_{\{0,1,2\}}\chi\left(F_{f|_{W\{0,1,2\}}},0\right) + c_{\{0,1,3\}}\left(F_{f|_{W\{0,1,3\}}},0\right) + c_{\{0,2,3\}}\chi\left(F_{f|_{W\{0,2,3\}}},0\right) + c_{\{1,2,3\}}\chi\left(F_{f|_{W\{1,2,3\}}},0\right) \end{array}\right)\right] \Big/ r_0 r_1 r_2 r_3.$$

Therefore,

$$\chi(F_{f,0}) = \frac{-492 - ((-6)(10) + (-6)(34) + (-8)(36) + 3(-8) + 4(-18))}{12}$$

$$= (-492 + 648)/12 = 156/12 = 13.$$

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