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Singular Values of One Parameter Family of Function $\frac{e^z+1}{2z}$

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Abstract

The singular values of one parameter family of functions $f_\lambda(z)=\lambda\frac{e^z+1}{2z}$, $\lambda\in R\setminus\{0\}, z\in C\setminus\{0\}$, are studied. It is shown that the function $f_\lambda(z)$ has infinitely many singular values. The critical values of $f_\lambda(z)$ lie exterior of the open disk and interior of the open disk according to two different regions.

Key words: Critical value, singular values.

1. Introduction

Studies on singular values are very important for the description of Juliasets, Fatou sets and other investigations in the complex dynamics. The dynamics of entire and geomorphic functions with infinitely many bounded or unbounded singular values are crucial to determine in comparison to that of functions with finitely many singular values; see Kapoor & Prasad (1998),Lim (2016), Nayak & Prasad (Jun 2010), Prasad (2005), Prasad & Nayak (2007), Sajid & Kapoor (2004). The singular values of one parameter family of functions are found by Sajid (2014 *a*,*b*, 2015 *a*,*b*,*c*). The singular values of transcendental geomorphic functions were also discussed by Zheng (2010).

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*)=0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w\in\hat{C}=C\cup\{\infty\}$ is said to be an asymptotic value for $f(z)$, if the reexists a continuous curve $\gamma:[0,\infty)\to\hat{C}$ satisfying $\lim\limits_{t\to\infty}\gamma(t)=\infty$ and $\lim\limits_{t\to\infty}f(\gamma(t))=w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically wounded or functions of bounded type if the set of all singular values of f is bounded; otherwise, it is called unbounded type.

Let

$$F=\left\{f_\lambda(z)=\lambda\frac{e^z+1}{2z}:\lambda\in R\setminus\{0\}, z\in C\setminus\{0\}\right\}$$

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Be one parameter family of transcendental functions. The function $f_\lambda \in F$ is neither even nor odd and not periodic. It has one pole at $z=0$.

This paper is organized as follows: In Theorem 2.1, it is found that the function $f_\lambda \in F$ has infinitely many singular values. It is shown that, in Theorem 2.2, the function $f'_\lambda(z)$ has no zeros in the right half plane. In Theorem 2.3, it is seen that the function $f_\lambda \in F$ maps two different regions exterior of the open disk and interior of the open disk centered at origin. Further, it is proved that the critical values of $f_\lambda(z)$ lie in the exterior of the open disk and interior of the open disk in Theorem 2.4.

2. Singular Values of $f_\lambda \in F$

In the following theorem, it is found that the function $f_\lambda \in F$ has infinitely many singular values:

Theorem 2.1. *Let $f_\lambda \in F$. Then, the function $f_\lambda(z)$ has infinitely many singular values.*

Proof. Since $f'_\lambda(z) = -\lambda \dfrac{(1-z)e^z+1}{2z^2}$, then the critical points of $f_\lambda(z)$ are solutions of the equation $(z-1)e^z-1=0$. Using real and imaginary parts of this equation, we have

$$\frac{y}{\sin y} + e^{y\cot y-1} = 0 \tag{1}$$

$$x = 1 - y\cot y \tag{2}$$

From Figure 1, it is observed that Equation (1) has infinitely many solutions, say $\{y_k\}_{k=-\infty,k\neq 0}^{k=\infty}$, since number of intersections increases when the size of interval increases on *x*-axis.

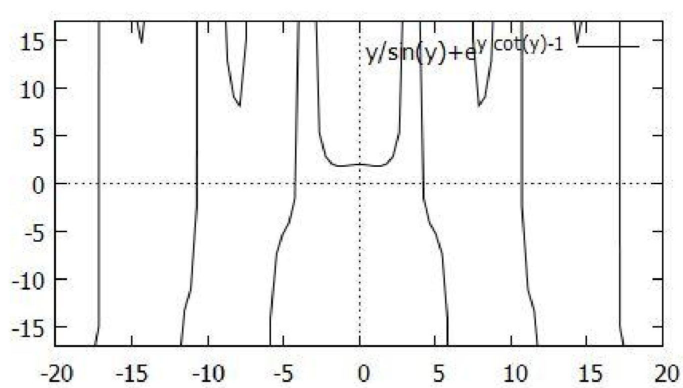

Figure 1: Graph of $\dfrac{y}{\sin y} + e^{y\cot y-1}$

From Equation (2), $x_k = 1 - y_k \cot y_k$ for $k = \pm1, \pm2, \pm3, \ldots\ldots$. Then, $z_k = x_k + iy_k$ are critical points of $f_\lambda(z)$. The critical values $f_\lambda(z_k) = \lambda \dfrac{e^{z_k} + 1}{2z_k}$ are distinct for k nonzero integers. It shows that the function $f_\lambda \in F$ has infinitely many critical values.

Since $f_\lambda(z) \to 0$ as $z \to \infty$ along negative real axis, it follows that the finite asymptotic value of $f_\lambda(z)$ is 0.

This proves that the function $f_\lambda \in F$ has infinitely many singular values. □

Let $H^+ = \left\{ z \in \hat{C} : \mathrm{Re}(z) > 0 \right\}$ and $H^- = \left\{ z \in \hat{C} : \mathrm{Re}(z) < 0 \right\}$ be the right half plane and left half plane respectively. The following theorem shows that the function $f'_\lambda(z)$ has no zeros in the right half plane:

Theorem 2.2. *Let $f_\lambda \in F$. Then, the function $f'_\lambda(z)$ has no zeros in the right half plane H^+ except one real positive zero.*

Proof. Since $f'_\lambda(z) = -\lambda \dfrac{(1-z)e^z + 1}{2z^2}$, then $e^{-z} = z - 1$. Writing in the real and imaginary parts, we have

$$e^{-x} \cos y = x - 1 \qquad (3)$$

$$e^{-x} \sin y = -y \qquad (4)$$

When $y = 0$, then $z = x > 0$ and, by Equation (3), $e^{-x} = x - 1$. For $x > 0$, it has only one real positive root.

When $y \neq 0$, then, by Equation (4), $\dfrac{\sin y}{y} = -e^x < -1$ for $x > 0$. This is not true for $y > 0$ since $\left| \dfrac{\sin y}{y} \right| < 1$. Moreover, since $\dfrac{\sin y}{y}$ is an even function, it is also false for $y < 0$.

Therefore, the function $f'_\lambda(z)$ has no zeros in H^+ except one real positive zero. □

Remark 2.1. $f'_\lambda(z)$ *has no zeros on imaginary axis since, from Equations (3) and (4),* $\cos y - i \sin y = 1 - iy$ *which gives* $y = 0$.

Suppose that the left half plane is divided in three regions $|z| < 1$, $1 \le |z| < 2$ and $|z| \ge 2$. The following theorem proves that the function $f_\lambda \in F$ maps two different regions exterior of the open disk and interior of the open disk centered at origin:

Theorem 2.3. *Let $f_\lambda \in F$. Then, the function $f_\lambda(z)$ maps the left half plane H^-*

(i) exterior of the open disk centered at origin and having radius $|\lambda|$ for $|z|<1$.

(ii) interior of the open disk centered at origin and having radius $|\lambda|$ for $|z| \ge 2$.

Proof. Suppose the function $h(z)=e^z$ for an arbitrary $z \in H^-$ and thel ine segment γ is defined by $\gamma(t)=tz, t\in[0,1]$. Then,

$$\int_\gamma h(z)\,dz = \int_0^1 h(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{tz}dt = e^z - 1$$

$$\left|e^z+1\right| = \left|\int_\gamma h(z)\,dz + 2\right| \tag{5}$$

Since $M \equiv \max_{t\in[0,1]}\left|h(\gamma(t))\right| = \max_{t\in[0,1]}\left|e^{tz}\right| < 1$ for $z \in H^-$, by Equation (5),

$$\left|e^z+1\right| \le M|z| + 2 < |z| + 2 < 2|z|$$

$$\left|\frac{e^z+1}{2z}\right| < 1 \text{ for all } |z| \ge 2.$$

It follows that $\left|f_\lambda(z)\right| = \left|\lambda\frac{e^z+1}{2z}\right| < |\lambda|$ for all $|z|\ge 2$. Therefore, the function $f_\lambda(z)$ maps the left half plane H^- interior of the open disk centered at origin and having radius $|\lambda|$ for $|z|\ge 2$.

Since $m \equiv \min_{t\in[0,1]}\left|h(\gamma(t))\right| = \min_{t\in[0,1]}\left|e^{tz}\right| > 0$ for $z \in H^-$, by Equation (5),

$$\left|e^z+1\right| \ge m|z| + 2 > 2 > 2|z|$$

$$\left|\frac{e^z+1}{2z}\right| > 1 \text{ for all } |z| < 1.$$

It shows that $\left|f_\lambda(z)\right| = \left|\lambda\frac{e^z+1}{2z}\right| > |\lambda|$ for all $|z|<1$. It gives that $f_\lambda(z)$ maps H^- exterior of the open disk centered at origin and having radius $|\lambda|$ *for* $|z|<1$. □

The following theorem shows that the critical values of $f_\lambda \in F$ lie in the exterior of the open disk and interior of the open disk according to mapping of two regions:

Theorem 2.4. *Let* $f_\lambda \in F$ *. Then, the critical values of* $f_\lambda(z)$ *lie*

(i) *exterior of the open disk centered at origin and having radius* $|\lambda|$ *for* $|z| < 1$.

(ii) *interior of the open disk centered at origin and having radius* $|\lambda|$ *for* $|z| \geq 2$.

Proof. Using Theorem 2.2, all the critical points of $f_\lambda \in F$ lie in H^- since $f'_\lambda(z)$ has no zeros in H^+ except one real positive zero. By Theorem 2.3, $f_\lambda \in F$ maps in H^- in the exterior of the open disk and interior of the open disk. The proof of theorem is completed. □

3. Conclusion

In this paper, we have described the singular values of the one parameter family $\frac{e^z+1}{2z}$. We have shown that the function $f_\lambda(z)$ has infinitely many singular values. Further, we have proved that the critical values of $f_\lambda(z)$ lie-in the exterior of the open disk and interior of the open disk.

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