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Lie Symmetry of Itô Stochastic Differential Equation Driven by Poisson Process

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Abstract

This work leads to an understanding of the random time change formulae for Poisson driven process in the context of Lie point symmetries without having to consult much of the intense Itôcalculus theory needed to formally derive it. We apply a form invariance methodology to derive the formula and apply it to a few examples.

Key words: Lie symmetries; Poisson process; stochastic differential equation.

1. Introduction

Lie symmetry theory of deterministic differential equations is well understood in literature [16, 17, 18, 19, 20] and can used for many important applications in the context of differential equations. For instance, for determination of group-invariant solutions, solving the first order differential equation, reducing the order of higher ODE, reducing the number of variables of partial differential equations and finding conservation laws.

In contrast to the deterministic differential equation, only a few attempts have been made to extend Lie group theory to the stochastic differential equation. It is worth noticing that the theory is still developing. Gaeta and Quintero [6] made the first approach to extend Lie symmetry of differential equations to Itô stochastic ordinary differential equations by which they consider a small class of transformations, i.e., fiber preserving transformations

$$\bar{x} = \theta_1(t, x, \epsilon), \qquad \bar{t} = \theta_2(t, \epsilon).$$

The method has been used to study the relationship between symmetries of stochastic systems to the symmetries of their corresponding Fokker-Planck equation. This is a restricted transformation that can only work to a fiber-preserving class of transformations which is a small sub-class of all possible transformations.

The second attempted [3, 4, 5, 8, 10, 15] succeed in applying symmetry transformations that include all the dependent variables in the transformation

$$\bar{x} = \theta_1(t, x, \epsilon), \qquad \bar{t} = \theta_2(t, x, \epsilon).$$

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This approach has been used to study the symmetry of scalar stochastic ordinary differential equations of first order [4] which reconciled the works of Meleshko S. V., Srihirun B. S. and Schultz E. [8] and Wafo Soh and F.M. Mahomed [10]. Furthermore, the formal method for finding Lie Point symmetries of scalar $\text{It}\hat{o}$ stochastic differential equations of the first order driven by the Wiener process was also discussed by E. Fredericks [3] with intention to correct and reconcile the finding of Srihirun and Schultz [8].

To the best of our knowledge in literature, all the methods above were applied only to the $\text{It}\hat{o}$ stochastic differential equations driven by Wiener processes [3-12]. In this paper we extend the Lie symmetry methods to the class of $\text{It}\hat{o}$stochastic differential equations driven by a Poisson process by implementing a more generalized $\text{It}\hat{o}$ formula and following the methodology of G. Gaeta [6] and E Fredericks and F. M. Mahomed [3].

We consider an $\text{It}\hat{o}$ stochastic differential equation driven by Poisson processes;

$$dX_i(t) = f_i\big(t, X(t)\big)\, dt + J_i\big(t, X(t)\big)\, dN(t) \tag{1.1}$$

with initial condition $X(0) = x_0$. So, equation (1.1) can be written in integral form as

$$X_i(t) = x_0 + \int_0^t f_i\big(s, X(s)\big)\, ds \; + \int_0^t J_i\,\big(s, X(s)\big)\, dN(s). \tag{1.2}$$

Where $f_i\big(t, X(t)\big)$ and $J_i\big(t, X(t)\big)$ are $n \times 1$ dimensional drift vector coefficients and Poisson diffusion coefficient respectively, which are assumed to satisfy Ikeda and Watanabe conditions for the uniqueness and existence of the solution of (1.1)while $dN(t)$ is the infinitesimal increment of the Poisson Process [12, 13, 14].

Symmetries of $(1.1)$are analysed by considering an infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} \; + \xi_i(t,x)\frac{\partial}{\partial x_i}. \tag{1.3}$$

The determining equations for $\text{It}\hat{o}$ stochastic differential equations (SDE) driven by Poisson processes (1.1) are derived using $\text{It}\hat{o}$calculus and are found to be non-stochastic.

Starting with an arbitrary function $F\big(t, X(t)\big)$which is once differential with respect to the spatial coordinate x and differentiable once with respect to temporal variable$t$, the $\text{It}\hat{o}$ Poisson diffusion process for $F\big(t, X(t)\big)$ of (1.1) exists [1, 2] and is

$$dF_j\big(t, X(t)\big) = \left(\frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i}\right)dt + \Big(F_j\big(t, X(t) + J_j\big(t, X(t)\big) - F_j\big(t, X(t)\big)\Big)\; dN(t). \tag{1.4}$$

The Einstein summation convention is assumed throughout this paper. Let

$$\Gamma_{(F)_j}\big(t, X(t)\big) = \frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i} \tag{1.5}$$

and

$$\Gamma^*_{(F)_j}\big(t, X(t)\big) = F_j\big(t, X(t) + J_j\big(t, X(t)\big) - F_j\big(t, X(t)\big) \tag{1.6}$$

Therefore (1.4) can be written as;

$$dF_j\big(t, X(t)\big) = \Gamma_{(F)_j}\big(t, X(t)\big)dt + \Gamma^*_{(F)_j}\big(t, X(t)\big)dN(t). \tag{1.7}$$

Using the It$\hat{o}$ multiplication properties of Poisson processes [1, 2]

$$dN(t) \cdot dN(t) = dN(t), \qquad dN(t) \cdot dt = 0 \text{ and } dt \cdot dt = 0$$

And application of infinitesimal transformations the determining equations for (SDE) with Poisson processes are derived and are non-stochastic. The main result can be summarised as

Theorem 1.1: The It$\hat{o}$ stochastic differential equation driven by Poisson processes

$$dX_i(t) = f_i\big(t, X(t)\big)\, dt + J_i\big(t, X(t)\big)\, dN(t) \tag{1.8}$$

Where $f_i\big(t, X(t)\big)$ and $J_i\big(t, X(t)\big)$ are the $n \times 1$ dimensional drift vector coefficient and the Poisson diffusion coefficient, with infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} \tag{1.9}$$

Has the following determining equations;

$$\left(f_j \Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)\big(t, X(t)\big) = 0, \tag{1.10}$$

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^*_{(\xi)_j}\right)\big(t, X(t)\big) = 0 \tag{1.11}$$

with additional conditions,

$$\Gamma^*_{(\tau)}\big(t, X(t)\big) = 0, \qquad and \quad \Gamma_{(\tau)}\big(t, X(t)\big) = c_1. \tag{1.12}$$

Where the operators $\Gamma\big(t, X(t)\big)$ and $\Gamma^*\big(t, X(t)\big)$are defined as in (1.5) and (1.6), and $\lambda > 0$ is called the intensity of the jump process or jump rate.

1. Lie Group Transformation

Consider a one parameter group of transformations of the time index t and the spatial variable x respectively,

$$\bar{t} = \theta_1(t, x, \epsilon\,), \qquad \bar{x} = \theta_2(t, x, \epsilon\,)$$

with the infinitesimals

$$\frac{\partial \theta_1}{\partial \epsilon} = \tau(t,x), \qquad \frac{\partial \theta_1}{\partial \epsilon} = \xi(t,x)$$

Satisfying the following initial conditions at $\epsilon = 0$

$$\bar{t}\big|_{\epsilon=0} = t, \qquad \bar{x}\big|_{\epsilon=0} = x.$$

A one parameter Lie group of infinitesimal transformations is therefore

$$\bar{t} = t + \epsilon\tau(t,x) + O(\epsilon) \tag{2.1}$$

And

$$\overline{X}(\bar{t}) = X(t) + \epsilon\, \xi(t,x) + O(\epsilon) \qquad (2.2)$$

Where ϵ is the parameter of the group, with the corresponding generator of the Lie algebra of the form

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}.$$

The differential point transformations of the spatial, temporal and the Poisson process variables are as follows

$$d\bar{t} = dt + \epsilon\, d\tau(t,x) + O(\epsilon), \qquad (2.3)$$

$$d\overline{X}(\bar{t}) = dX(t) + \epsilon\, d\xi(t,x) + O(\epsilon) \qquad (2.4)$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\big(\lambda\, dt + dN(t)\big) + O(\epsilon). \qquad (2.5)$$

Using the It$\hat{o}$formula(1.7), we have the spatial and temporal infinitesimals in It$\hat{o}$forms as

$$d\xi_j = \Gamma_{(\xi)_j}\, dt + \Gamma^*_{(\xi)_j}\, dN(t) \qquad (2.6)$$

$$d\tau = \Gamma_{(\tau)}\, dt + \Gamma^*_{(\tau)}\, dN(t) \qquad (2.7)$$

where $\Gamma_{(\xi)_j}, \Gamma^*_{(\xi)_j}, \Gamma_{(\tau)}\ and\ \Gamma^*_{(\tau)}$are the drift and diffusion coefficients of the spatial and temporal infinitesimals, respectively defined using the operators (1.5) and (1.6).

By substitution of the infinitesimal of spatial (2.6) and temporal variables (2.7) in (2.3), (2.4) and (2.5), and also using the It$\hat{o}$ multiplication properties we proceed to get the group transformations of temporal, spatial and jump variables in It$\hat{o}$ forms

$$d\bar{t} = dt + \epsilon\left(\Gamma_{(\tau)}\, dt + \Gamma^*_{(\tau)}\, dN(t)\right) + O(\epsilon), \qquad (2.8)$$

$$d\bar{x} = dx + \epsilon\left(\Gamma_{(\xi)_j}\, dt + \Gamma^*_{(\xi)_j}\, dN(t)\right) + O(\epsilon) \qquad (2.9)$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{\Gamma_{(\tau)}\, dt + \Gamma^*_{(\tau)}\, dN(t)}{dt}\big(\lambda\, dt + dN(t)\big) + O(\epsilon). \qquad (2.10)$$

Expending the It$\hat{o}$ infinitesimal of the jump variable (2.10) by utilising the Poisson process differential multiplication properties we get

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\left(\lambda\left(\Gamma_{(\tau)}\, dt + \Gamma^*_{(\tau)}\, dN(t)\right) + \Gamma_{(\tau)}\, dN(t) + \Gamma^*_{(\tau)}\frac{dN(t)}{dt}\right) + O(\epsilon). \qquad (2.11)$$

1.1 Invariance form of the Spatial Process

To ensures the recovery of the finite transformations from the infinitesimal transformation, we need to transform $dX_j(t)$ into

$$d\overline{X}_j(\bar{t}) = \overline{f_j}\big(t, X(t)\big)\, d\bar{t} + \overline{J_j}\big(\bar{t}, \overline{X}(\bar{t})\big)\, d\overline{N}(\bar{t}) \quad (2.12)$$

where the transformed drift component $\overline{f_j}\big(t, X(t)\big)$ and jump component $\overline{J_j}\big(\bar{t}, \overline{X}(\bar{t})\big)$ using the infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i},$$

respectively are

$$\overline{f_i}\big(t, X(t)\big) = \Big(f_j + \epsilon\, H\big(f_j\big)\Big)\big(t, X(t)\big)$$

$$= f_j\big(t, X(t)\big) + \epsilon\left(\tau\frac{\partial f_j}{\partial t} + \xi_i\frac{\partial f_j}{\partial x_i}\right)\big(t, X(t)\big) \quad (2.13)$$

and

$$\overline{J_j}\big(\bar{t}, \overline{X}(\bar{t})\big) = \Big(J_j + \epsilon\, H\big(J_j\big)\Big)\big(t, X(t)\big)$$

$$= J_j\big(t, X(t)\big) + \epsilon\left(\tau\frac{\partial J_j}{\partial t} + \xi_i\frac{\partial J_j}{\partial x_i}\right)\big(t, X(t)\big) \quad (2.14)$$

1.2 Poisson Invariance Properties

We apply the invariance to the moments of the Poisson process to ensure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Poisson process which are:

$$E_Q[dN(t)] = \lambda \cdot dt \quad (2.15)$$

$$E_Q[dN(t) \cdot dN(t)] = \lambda \cdot dt. \quad (2.16)$$

The invariance of the instantaneous mean of the transformed Poisson process under new measure $\bar{Q}$ is

$$E_{\bar{Q}}\big[d\overline{N}(\bar{t})\big] = \lambda \cdot d\bar{t} \quad (2.17)$$

Expanding (2.17) using the It$\hat{o}$ forms of jump (2.8) and temporal group transformations (2.11) we get

$$\Gamma^*_{(\tau)}\big(t, X(t)\big) = 0 \quad (2.18)$$

Next, we apply the invariance form to instantaneous variance of the transformed Poisson process measure (2.16) from which using (2.11) we have

$$E_{\bar{Q}}\big[d\overline{N}(\bar{t}) \cdot d\overline{N}(\bar{t})\big] = \lambda \cdot d\bar{t} \quad (2.19)$$

Thus, using (2.18) and the It$\hat{o}$temporal group transformation (2.8) we have derived the following generalized random time change formula

$$\bar{t} = \int^{t} \Gamma_{(\tau)}(s)ds \quad (2.20)$$

With

$$\Gamma_{(\tau)}\big(t, X(t)\big) = constant = c_1 \quad (2.21)$$

Using the probabilistic invariance property of the transformed time index differential, i.e.,

$$E_{\bar{Q}}[d\bar{t}] = d\bar{t}. \quad (2.22)$$

Finally, we can conclude from (2.18) the temporal infinitesimal $\tau(t,x)$does not depend on x, therefore $\tau(t,x) = \tau(t)$.

Definition 2.1 The infinitesimal transformations (2.3) and (2.4) i.e.,

$$\bar{t} = t + \epsilon\tau(t,x) + O(\epsilon), \quad \bar{X} = X + \epsilon\xi(t,x) + O(\epsilon) \quad (2.23)$$

Are called Lie symmetry transformations of (1.1) if they leave the It$\hat{o}$ stochastic differential equation (1.1)

$$dX_i(t) = f_i\big(t, X(t)\big)\, dt + J_i\big(t, X(t)\big)\, dN(t) \quad (2.24)$$

And the infinitesimal moments for the differential Poisson process i.e.,

$$E_Q[dN(t)] = \lambda \cdot dt, \quad E_Q[dN(t) \cdot dN(t)] = \lambda \cdot dt \text{ and } E_Q[dt] = dt \quad (2.25)$$

Invariant. Where $\lambda > 0$ and ϵ is the jump intensity and group parameter respectively.

2. Determining Equations

In this section, we will derive the determining equations for the admitted symmetries of (1.1). The intention is to transform $d\bar{X}_j(\bar{t})$ into

$$d\bar{X}_j(\bar{t}) = \bar{f}_j\big(t, X(t)\big)\, d\bar{t} + \bar{J}_j\big(\bar{t}, \bar{X}(\bar{t})\big)\, d\bar{N}(\bar{t}) \quad (3.1)$$

Substituting the transformed drift coefficient (2.13), Poisson vector coefficients(2.14), It$\hat{o}$forms of temporal (2.8) and Poisson group transformation (2.11) into (3.1) we get

$$d\bar{X}_j(\bar{t}) = dX_j(t) + \epsilon\left(\bar{f}_j\Gamma_{(\tau)}\big(t, X(t)\big) + \frac{\lambda J_j}{2}\Gamma_{(\tau)}\big(t, X(t)\big) + H(f_j)\right)dt$$

$$+\left(\Gamma^*_{(\tau)}\big(t, X(t)\big) + \frac{J_j}{2}\Gamma_{(\tau)}\big(t, X(t)\big) + H(J_j)\right)dN(t) \quad (3.2)$$

Therefore, by comparing transformed stochastic differential equation (3.2) and the It$\hat{o}$ form of the spatial group transformation (2.9) we have the following determining equations

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)\big(t, X(t)\big) = 0, \quad (3.3)$$

and

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^*_{(\xi)_j}\right)\big(t, X(t)\big) = 0. \quad (3.4)$$

The invariance of the instantaneous mean of the transformed differential Poisson process (2.17) gives additional conditions i.e., from (2.18) and (2.21) we get

$$\Gamma^*_{(\tau)}\big(t, X(t)\big) = 0 \; and \; \Gamma_{(\tau)}\big(t, X(t)\big) = c_1. \tag{3.5}$$

Equation (3.3) can be interpreted using the definition of first prolongation of an infinitesimal generator for non-stochastic ordinary differential equations as follows

$$H^{[1]} = H + \eta_i^{[1]} \frac{\partial}{\partial \dot{x}_\iota}. \tag{3.6}$$

Where

$$\dot{x}_\iota = \frac{dx_i}{dt} = D_t \, x_i \tag{3.7}$$

and

$$\eta_i^{[1]} = D_t(\xi_i) - \dot{x}_\iota D_t(\tau) \tag{3.8}$$

$$= \frac{\partial \xi_i}{\partial t} + \dot{x}_\iota \frac{\partial \xi_i}{\partial x} - \dot{x}_\iota \left(\frac{\partial \tau}{\partial t} + \dot{x}_\iota \frac{\partial \tau}{\partial x}\right) \tag{3.9}$$

with total time derivative D_t defined as

$$D_t = \frac{\partial}{\partial t} + \dot{x}_\iota \frac{\partial}{\partial x} + \ddot{x}_\iota \frac{\partial}{\partial \dot{x}_\iota} + \cdots \tag{3.10}$$

Using the definition of first prolongation on $(\dot{x}_\iota - f_i)$ at $\dot{x}_\iota = f_i$, can be expressed as

$$H^{[1]}(\dot{x}_\iota - f_i)_{\,\dot{x}_\iota = f_i} = \eta_i^{[1]} - H(f_i). \tag{3.11}$$

Using (3.8) and (3.11) equation (3.4) can be written as

$$H^{[1]}(\dot{x}_\iota - f_i)_{\,\dot{x}_\iota = f_i} - \frac{\lambda J_j}{2} \Gamma_{(\tau)} \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x}\right) = 0. \tag{3.12}$$

Where the operators $\Gamma_{(\tau)}\big(t, X(t)\big)$, $\Gamma^*_{(\tau)}\big(t, X(t)\big)$ are defined in (1.5), (1.6) respectively, and λ is called the jump rate or jump intensity of the Poisson process.

Remark 3.1: The extra condition obtained from the invariance of the instantaneous mean of the transformed differential Poisson process (2.17) forces the temporal infinitesimal $\tau(t, x)$ to be a function of the time variable only. This implies that we are now dealing with a fiber-preserving infinitesimal generator i.e.,

$$H = \tau(t) \frac{\partial}{\partial t} + \xi_i(t, x) \frac{\partial}{\partial x_i}. \tag{3.13}$$

3. Applications

In this section, we are going to apply the derived determining equations of Poisson It$\hat{o}$ stochastic differential equations obtained in the previous section to some Poisson process models to show how the determining equations can be used to find the admitted Lie point symmetries of each model.

Example 4.1: Consider the Poisson SDE, linear in the state process $X(t)$, with constant coefficients,

$$dX(t) = X(t)\big(u_0(t)\,dt + v_0(t)\,dN(t)\big) \quad (4.1)$$

With initial condition $X(t_0) = x_0 > 0$, $u_0(t) = 2$ is called the drift or deterministic coefficient and $v_0(t) = 1$ is the jump amplitude coefficient of the jump term, with jump rate$\lambda = \lambda_0$.

Using the determining equations (3.3) and (3.4)respectively we have

$$\left(2x\Gamma_{(\tau)} + \lambda_0 x\frac{\Gamma_{(\tau)}}{2} + 2\xi(t,x) - \Gamma_{(\xi)}\right)\big(t, X(t)\big) = 0 \quad (4.2)$$

$$2x\frac{\partial\tau(t)}{\partial t} + \frac{\lambda_0 x}{2}\frac{\partial\tau(t)}{\partial t} + 2\xi(t,x) - \frac{\partial\xi(t,x)}{\partial t} - 2x\frac{\partial\xi(t,x)}{\partial x} = 0 \quad (4.3)$$

and

$$\left(x\frac{\Gamma_{(\tau)}}{2} + \xi(t,x) - \Gamma^*_{(\xi)}\right)\big(t, X(t)\big) = 0 \quad (4.4)$$

$$\frac{x}{2}\frac{\partial\tau(t)}{\partial t} + \xi(t, x + x) + \xi(t,x) = 0. \quad (4.5)$$

Using (2.18) and (2.21) we get the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. (4.6)$$

Substituting the temporal infinitesimal (4.6) in (4.3) and (4.5) respectively gives

$$\frac{c_1 x(\lambda_0 + 4)}{2} + 2\xi(t,x) - \frac{\partial\xi(t,x)}{\partial t} - 2x\frac{\partial\xi(t,x)}{\partial x} = 0(4.7)$$

and

$$\frac{c_1 x}{2} + 2\xi(t,x) - \xi(t,2x) = 0. (4.8)$$

Differentiating (4.7) with respect to x gives

$$\frac{c_1(\lambda_0 + 4)}{2} + 2\frac{\partial\xi(t,x)}{\partial x} - \frac{\partial^2\xi(t,x)}{\partial t\partial x} - 2\frac{\partial\xi(t,x)}{\partial x} - 2x\frac{\partial^2\xi(t,x)}{\partial x^2} = 0. (4.9)$$

Differentiating (4.8) with respect to x gives

$$\frac{c_1}{2} + 2\frac{\partial\xi(t,x)}{\partial x} - 2\frac{\partial\xi(t,2x)}{\partial x} = 0. \quad (4.10)$$

Differentiating (4.10) with respect to t gives

$$\frac{\partial^2\xi(t,x)}{\partial t\partial x} = \frac{\partial^2\xi(t,2x)}{\partial t\partial x}. \quad (4.11)$$

Equation (4.11) implies

$$\frac{\partial^2\xi(t,x)}{\partial t\partial x} = \frac{\partial^2\xi(t)}{\partial t\partial x} = \frac{df(t)}{dt}\ . \tag{4.12}$$

Solving the differential equation (4.12) we get

$$\xi(t,x) = f(t)x + g(x). \tag{4.13}$$

By substituting (4.13) into (4.9) we get

$$\frac{c_1(\lambda_0 + 4)}{2} = \frac{df(t)}{dt} + 2x\frac{d^2g(x)}{dx^2}. \tag{4.14}$$

When differentiating (4.14) with respect to time we get

$$\frac{d^2f(t)}{dt^2} = 0. \tag{4.15}$$

Solving the ordinary differential equation (4.15) implies $f(t)$ is linear in t i.e.,

$$f(t) = c_3t + c_4. \tag{4.16}$$

After substituting (4.16) into (4.13) we arrive at the spatial infinitesimal

$$\xi(t,x) = (c_3t + c_4)x + g(x). \tag{4.17}$$

Substituting (4.17) into (4.14) results in

$$\frac{c_1(\lambda_0 + 4)}{2} = c_3 + 2x\frac{d^2g(x)}{dx^2}, \tag{4.18}$$

Which implies that

$$\frac{d^2g(x)}{dx^2} = \frac{\frac{c_1(\lambda_0+4)}{2} - c_3}{2x}. \tag{4.19}$$

Solving the differential equation (4.19) for $g(x)$ finally gives

$$g(x) = \frac{\frac{c_1(\lambda_0+4)}{2} - c_3}{2}(xln|x| - x) + c_5x + c_6, \tag{4.20}$$

Therefore, using (4.20) the special infinitesimal (4.17) can be written as

$$\xi(t,x) = (c_3t + c_4)x + \frac{\frac{c_1(\lambda_0+4)}{2} - c_3}{2}(xln|x| - x) + c_5x + c_6. \tag{4.21}$$

However, substituting (4.21) in (4.8) we have

$$\frac{c_1 x}{2} + 2\left((c_3 t + c_4)x + \frac{\frac{c_1(\lambda_0+4)}{2} - c_3}{2}(xln|x| - x) + c_5 x + c_6 \right)$$
$$= 2(c_3 t + c_4)x + \frac{\frac{c_1(\lambda_0+4)}{2} - c_3}{2}(2xln|2x| - 2x) + 2c_5 x + c_6. \qquad (4.22)$$

Which can be simplified to get

$$\frac{c_1 x}{2} + c_6 = \frac{\frac{c_1(\lambda_0+4)}{2} - c_3}{2}(xln|4|). \qquad (4.23)$$

Further comparison of the coefficients of powers of x in (4.23), gives

- $x : c_3 = c_1\left(\frac{(\lambda_0+4)}{2} - \frac{1}{ln|4|}\right)$ and
- $x^0 : c_6 = 0.$

Thus, the spatial infinitesimal (4.21) finally becomes

$$\xi(t,x) = c_1\left(\left(\frac{ln|4|(\lambda_0+4) - 2}{ln|16|}\right)tx + \frac{xln|x| - x}{ln|16|}\right) + c_4 x + c_5 x. \qquad (4.24)$$

So we have three symmetry generators corresponding to the infinitesimals

$$H_1 = t\frac{\partial}{\partial t} + \left(\left(\frac{ln|4|(\lambda_0+4) - 2}{ln|16|}\right)tx + \frac{xln|x| - x}{ln|16|}\right)\frac{\partial}{\partial x}, \quad H_2 = \frac{\partial}{\partial t}, \quad H_3 = 2x\frac{\partial}{\partial x}. \quad (4.25)$$

The infinitesimal generators (4.25) give the following Lie bracket relations in *Table 1* below

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-H_4$	$-\frac{H_3}{ln\|16\|}$
H_2	H_4	0	0
H_3	$\frac{H_3}{ln\|16\|}$	0	0

Table 1: Commentator table for the Lie algebra generators (4.25)

The commentator table shows that the infinitesimals generators (4.25) is closed under Lie bracket relations and hence is a Lie algebra, where H_4 is linear combination of H_3 and H_2 given as

$$H_4 = \alpha H_3 + H_2 \;\; with\, \alpha = \frac{ln|16| - 1 + ln|2|\lambda_0}{ln|16|}. \qquad (4.26)$$

Example 4.2: Consider a Poisson driven stochastic differential equation

$$dX(t) = -kt^2 dt + b\, dN(t) \;\; with\; b \neq 0 \quad (4.27)$$

And initial condition $X(0) = x_0$.

Using the determining equations (3.3) and (3.4)respectively we have

$$-kt^2\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x}\right) + \frac{b\,\lambda}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x}\right) - 2kt\tau(t,x) = \frac{\partial\xi}{\partial t} - kt^2\frac{\partial\xi}{\partial x} \quad (4.28)$$

and

$$\frac{b}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x}\right) = \xi(t, x+b) - \xi(t,x). \quad (4.29)$$

Using equation (2.18) and (2.21) we get the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. \quad (4.30)$$

Using temporal infinitesimal (4.30) in (4.28) and (4.29) we respectively have

$$c_1\left(\frac{b\,\lambda}{2} - kt^2\right) - 2kt(c_1 t + c_2) = \frac{\partial\xi}{\partial t} - kt^2\frac{\partial\xi}{\partial x} \quad (4.31)$$

And

$$\xi(t, x+b) - \xi(t,x) = \frac{b\,c_1}{2}. \quad (4.32)$$

Differentiating (4.31) and (4.32) with respect to x respectively gives

$$\frac{\partial^2\xi}{\partial t\partial x} - kt^2\frac{\partial^2\xi}{\partial x^2} = 0 \quad (4.33)$$

and

$$\frac{\partial\xi(t, x+b)}{\partial x} = \frac{\partial\xi(t,x)}{\partial x}. \quad (4.34)$$

Equation (4.34) implies

$$\frac{\partial\xi(t,x)}{\partial x} = \frac{\partial\xi(t)}{\partial x} = f(t). \quad (4.35)$$

Differentiating (4.35) with respect to x gives

$$\frac{\partial^2\xi}{\partial x^2} = 0, \quad (4.36)$$

Solving the differential equation (4.36) we have

$$\xi(t,x) = f(t)x + g(t). \tag{4.37}$$

Substituting (4.37) into (4.33) implies

$$\frac{df(t)}{dt} = 0. \tag{4.38}$$

Equation (4.38) implies $f(t)$ is constant i.e.,

$$f(t) = c_3, \tag{4.39}$$

Therefore, using (4.39) and (4.37) we have

$$\xi(t,x) = c_3 x + g(t). \tag{4.40}$$

Substituting (4.40) into (4.32) gives this relation

$$c_1 = 2c_3. \tag{4.41}$$

Using (4.40) and (4.41), equation (4.31) gives

$$c_1\left(\frac{b\,\lambda}{2} - 3kt^2\right) - 2ktc_2 = \frac{dg(t)}{dt} - \frac{kt^2 c_1}{2}. \tag{4.42}$$

Solving the differential equation (4.42) gives

$$g(t) = c_1\left(\frac{b\lambda t}{2} - \frac{5kt^3}{6}\right) - kt^2 c_2 + c_4. \tag{4.43}$$

Therefore, substituting (4.43) into (4.40) the spatial infinitesimal finally becomes

$$\xi(t,x) = c_1\left(\frac{b\lambda t}{2} - \frac{5kt^3}{6} + \frac{x}{2}\right) - kt^2 c_2 + c_4. \tag{4.44}$$

Finally, the Poisson diffusion model admitted three dimensional symmetry infinitesimal generators;

$$H_1 = t\frac{\partial}{\partial t} + \left(\frac{b\lambda t}{2} - \frac{5kt^3}{6} + \frac{x}{2}\right)\frac{\partial}{\partial x}, \quad H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \quad H_3 = \frac{\partial}{\partial x}. \tag{4.45}$$

With the corresponding Lie bracket relations of the generators (4.45) given in *Table 2* as

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-H_4$	$-\frac{H_3}{2}$
H_2	H_4	0	0
H_3	$\frac{H_3}{2}$	0	0

Table 2: Commentator table for the Lie algebra generators (4.45)

The Lie bracket relations in *Table 2* above show that the infinitesimal generator (4.45) satisfied Lie commutative relation properties and hence forms a Lie algebra, where

$$H_4 = H_2 - \frac{b\lambda}{2} H_3 \text{ is the linear combination of } H_2 \text{ and } H_3.$$

4. Conclusion

Lie Symmetry analysis for Itô stochastic differential equations driven the by Poisson processes was carried out, infinitesimals of the Poisson process $dN(t)$ were derived using the moments invariance properties of the process. Determining equations were derived and found to be deterministic even though they describe stochastic differential equation. Examples are given to show how the determining equations can be used to find the symmetries, symmetries admitted by (1.1) are found to be fiber-preserving symmetries. Finally, the Lie bracket relation was obtained which shows that all the infinitesimal generators found are closed under the Lie bracket and hence they form a Lie algebra. Classification of the given examples is presented in *Table 3*.

Group Dimension	Basis Operators	Equations
3	$H_1 = t\frac{\partial}{\partial t} + \left(\left(\frac{ln\|4\|(\lambda_0 + 4) - 2}{ln\|16\|}\right)tx + \frac{xln\|x\| - x}{ln\|16\|}\right)\frac{\partial}{\partial x}$, $H_2 = \frac{\partial}{\partial t}$, $H_3 = 2x\frac{\partial}{\partial x}$.	$dX(t) = X(t)\big(2\,dt + dN(t)\big)$
3	$H_1 = t\frac{\partial}{\partial t} + \left(\frac{b\lambda t}{2} - \frac{5kt^3}{6} + \frac{x}{2}\right)\frac{\partial}{\partial x}$, $H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}$, $H_3 = \frac{\partial}{\partial x}$	$dX(t) = -kt^2dt + b\,dN(t)$, $b \neq 0$

Table 3: Lie Group Classification

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