Generalizedextended Weibull Power Series Family of Distributions

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Abstract

In this study, we introduce a new familyof models for lifetime data called generalized extended Weibullpower series family of distributions by compoundinggeneralizedextended Weibull distributions and power series distributions. The compounding procedure follows the same setup carried out by Adamidis (1998). The proposed family contains all types of combinations between truncated discrete with generalized and non-generalized Weibull distributions. Some existing power series and subclasses of mixed lifetime distributions become special cases of the proposed family, such as the compound class of extended Weibull power seriesdistributions proposed by Silva et al. (2013) and the generalized exponential power series distributionsintroduced by Mahmoudi and Jafari (2012).Some mathematical properties of the new class are studied, includingthe cumulative distribution function, density function, survival function, and hazard rate function. The method of maximum likelihood is used for obtaining a general setup for estimating the parameters of any distribution in this class. An expectation-maximization algorithm is introduced for estimating maximum likelihood estimates.Special subclasses and applications for some models in areal datasetare introduced to demonstrate the flexibility and the benefit of this new family.

Keyword: Generalized extended Weibull power series distributions, Weibull power series distribution, generalized power series distributions

1. Introduction

The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields, such as public health, actuarial science, biomedical studies, demography, and industrial reliability. In risk modeling, the lifetime associated with a particular risk is not observable, asonly the maximum or the minimum lifetime value among all the risks can be observed.In reliability, we observe only the maximum component lifetime of a parallel system and the cause of failure. Lifetime data modeling is introduced by compounding any continuous distribution and power series distributions. The Weibull distribution is exhaustively used for describing hazard rates due to its negatively and positively skewed density shapes. Chahkandi and Ganjali (2009)proposed the exponential power series family of distributionsthatgeneralizeinto a two-parameter exponential power seriescalled the Weibull power series (WPS) class of distributions by Morais and Barreto-Souza (2011). The WPSdistributions can have an increasing, decreasing, and upside-down bathtub failure rate function. In the same manner, the exponentiated Weibull power series distribution and itsapplications were presented by Mahmoudi and Shiran (2012). Recently, the exponentiatedWeibull–Poisson distribution and its applications wereintroduced by Mahmoudi and Sepahdar (2013). The generalized exponential power series (GEPS) distributions were proposed by Mahmoudi and Jafari (2012),following the same approach developed by Morais and Barreto-Souza (2011) by compounding the generalized exponential and the power series distributions. Leahu et al. (2013) proposed the power series distributions in the lifetime as the maximum or minimum of the sample with a power series distributed size.

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The complementary exponential power series distribution with increasing failure rate was introduced by Jose et al. (2013) as a complementto the exponential power series model proposed by Chahkandy and Ganjali (2009). Silva et al. (2013)introduced thepower series distributions of the compound class of the extended Weibull distribution. Recently, Bourguignon et al. (2014) proposed a new class of fatigue life distribution as the Birnbaum –Saunders power series class of distributions. We combine the GEPS distributions introduced byMahmoudi and Jafari (2012)and the compound class of the extended Weibull power series distributions (EWPS) proposed by Silva et al. (2013) into a more general family called the generalized extended Weibullpower series (GEWPS).

Consider a system with N components, where N (the number of components) is a discrete random variable with support $\{1, 2, ...\}$. The lifetime of the ith ($1, 2, ..., N$) component is the nonnegative continuous random variable, say X_i, thedistribution of which belongs to one of the lifetime distributions, such as exponential, gamma, Weibull, and Pareto, among others. The discrete random variable N can have severaldistributions, such as zero-truncated Poisson,geometric, binomial, logarithmic, and the power series distributions in general. The non-negative random variable X denoting the lifetime of such a system is defined by $X = \min\{X_i\}$ or $X = \min\{X_i\}$ based on whether the components are series or parallel.By taking a system with parallel components in which the random variable N has the power series distributions and the random variable X_i follows the generalized Weibulldistribution, we introduce the GEWPS class of distributions that contain the GEPSand the EWPS distributions as special cases.

This study aims to generalize the EWPS distributions to obtain a new and more flexible family to describe reliability data. The proposed family can be applied to other fields, including business, environment, actuarial science, biomedical studies, demography and industrial reliability,and many other fields. This family contains several subclasses and lifetime models as special cases.

This paper is organized as follows. In Section 2, we define the class of Weibull and generalized Weibull distributions, and demonstrate the many existing models that can be deduced as special cases of the proposed unified model. In Section 3, we define the GEWPS class of distributions in terms of distribution functions and special cases of some existing classes. In Section 4, we provide the general properties of the GEWPS class, including the densities and thesurvival and hazard rate functions. Quantiles, moments, and order statistics of GEWPS are discussed in Section 5. The estimation of the GEWPS parameters is investigated in Section 6 using the maximum likelihood method withexpectation-maximization(EM) algorithm and a large sample inference. In Section 7, special subclasses and some special distributions are introduced along with the flexible mathematical forms of theirproperties. In Section 8, two models are presented and applied to illustrate how to use the proposed family.Finally, some concluding remarks are addressed in Section 9.

2. The class of Weibull and Generalized Weibull distribution

Weibull distribution is one of the most widely used lifetime distributions in terms of reliability. A large number of modifications have been suggested for the Weibull distributionto improve the shape of the hazard rate function. Pengand Yan (2014) presented many references on this matter. The class of extended Weibull distributions (EW) was proposed by Gurvich et al. (1997). This class is illustrated by the following definition.

Definition 1: Arandom variable X_i is a member of the Weibull class of distributionif its cumulative distribution function (cdf) is given by

$$G_W(x; \alpha, \xi) = G_W(x) = 1 - e^{-\alpha H(x; \xi)}; \ x, \alpha > 0, \quad (1)$$

where $H(x; \xi) = H(x)$ is a non-negative monotonically increasing function that depends on the parameter vector $\xi > 0$. The corresponding probability density function (pdf) becomes

$$g_W(x; \alpha, \xi) = g_W(x) = \alpha h(x) e^{-\alpha H(x)}; \ x, \alpha > 0, \quad (2)$$

Where $h(x) = h(x;\xi)$ is the first derivative of $H(x)$. The distributions of most Weibull typescan be re-written in form (1) depending on the choice of the function $H(x)$. Silva et al. (2013) and Gurvich et al. (1997)listed some examples for this class. By using the idea of Gupta and Cundu (1999), the generalized exponential of this class can be modified as follows:

Definition 2: Arandom variable X_i belongs to the generalized extended Weibull distribution class if its cdf is given by

$$G(x;\alpha,\delta,\xi) = G(x) = (1 - e^{-\alpha H(x)})^{\delta}; \quad x, \alpha, \delta > 0, (3)$$

where $H(x)$ is a non-negative monotonically increasing function thatdepends on a parametervector $\xi > 0$. The corresponding pdf becomes

$$g(x;\alpha,\delta,\xi) = g(x) = \alpha\delta h(x)e^{-\alpha H(x)}(1 - e^{-\alpha H(x)})^{\delta-1}; \quad x, \alpha, \delta > 0, (4)$$

where $h(x)$ is the first derivative of $H(x)$. One can see that $G(x) = (G_W(x))^{\delta}$, and thus $g(x) = \delta(G_W(x))^{\delta-1}g_W(x)$. The distributions of most Weibull and exponentiated Weibull typescan be written in form (3) depending on the choice of the function $H(x)$ and δ. Table 1 displays useful $H(x)$ and corresponding parameter vectorsfor some existing distributions.

Table 1: Special distributions and the corresponding $H(x;\xi)$ and vector ξ.

Distribution	$H(x)$	α	δ	ξ	References
Exponential	x	α	1	-	Johnson et al.(1994)
Exponential power	$e^{(\lambda x)^{\beta}-1}$	α	1	$[\lambda,\beta]$	Smith and Bain (1975)
Burr XII $(x \geq 0)$	$\log(1+x^c)$	α	1	c	Rodriguez (1977)
Weibull $(x \geq 0)$	x^{γ}	α	1	γ	Johnson et al.(1994)
Modified Weibull	$x^{\gamma}e^{\lambda x}$	α	1	$[\gamma,\lambda]$	Lai et al. (2003)
Weibull extension	$\lambda[e^{(x/\lambda)^{\beta}} -1]$	α	1	$[\lambda,\beta]$	Xie et al. (2002)
Exponential power	$\exp[(\lambda x)^{\beta}]-1$	α	1	$[\beta,\lambda]$	Smith and Bain (1975)
Pareto $(x \geq k)$	$\log(x/k)$	α	1	k	Johnson et al.(1994)
Generalized exponential	x	α	δ	-	Gupta and Kundu (2000)
ExponentiatedWeibull	$\alpha^{\gamma-1}x^{\gamma}$	α	δ	γ	Nassar and Eissa (2003)
Exp. Mod. Weibull ext.	$\lambda[1-e^{(x/\lambda)^{\beta}}]$	α	δ	$[\lambda,\beta]$	Sarhan and Apaloo (2013)
Exponentiated Rayleigh	x^2	α	δ	-	Surles and Padgett (2001)

3. The GEWPS family

In this section, we derive the family of GEWPS distributions by compounding the generalized extended Weibull class and power series distributions.

Let N, be a zero truncated discrete random variable having a power series distribution with the following probability mass function:

$$p_n = p(N = n) = \frac{a_n \theta^n}{c(\theta)}, \quad n = 1, 2, ...,$$

$$(5)$$

where $a_n \geq 0$ depends only on n, $c(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$, and $\theta \in (0, s)$ is chosen in such a way that $c(\theta)$ is finite. Given N, let $X_1, \cdots, X_N$ be independent and identically distributed (iid) random variables following (3). Let $X_{(n)} = \max\{X_i\}_{i=1}^{N}$. Then, the cdf of $X_{(n)} | N = n$ is given by

$$G_{X(n)|N=n}(x) = [1 - e^{-\alpha H(x)}]^{n\delta}, \quad x > 0, n \geq 1.$$

That is, $X_{(n)} | N = n$ has a generalized extended Weibull class of distribution with parameters θ, δ, α and ξ based on the same function $H(x)$. The generalized power series distribution denoted by GEWPS with an increasing failure rate is defined by the marginal distribution (cdf) of $X_{(n)}$:

$$F_{X_{(n)}}(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{c(\theta)} (1 - e^{-\alpha H(x)})^{\delta n} = \frac{c(\theta(1 - e^{-\alpha H(x)})^\delta)}{c(\theta)}, \quad x > 0,$$

which can be written as

$$F_{X_{(n)}}(x) = \sum_{n=1}^{\infty} p_n (G(x))^n = \frac{c(\theta G(x))}{c(\theta)}, \quad x > 0.$$

$$(6)$$

Note that if $H(x) = x$, the model is reduced to the GEPS introduced by Mahmoudi and Jafari (2012).

Remarks. Let $X_{(1)} = \min\{X_i\}_{i=1}^{N}$, then

(1)   The cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x) = 1 - \frac{c(\theta - \theta G(x))}{c(\theta)} = 1 - \frac{c(\theta(1 - G(x)))}{c(\theta)}.$$

$$(7)$$

Note that if $\delta = 1$, then the

cdf of $X_{(1)}$ is:

$$F_{X_{(1)}}(x) = 1 - \frac{c(\theta(1 - G_W(x)))}{c(\theta)} = 1 - \frac{c(\theta e^{-\alpha H(x)})}{c(\theta)},$$

which is called the EWPS proposed by Silva et al. (2013).

If $\delta = 1$ and if $H(x) = x$, then the cdf of $X_{(1)}$ becomes

$$F_{X_{(1)}}(x) = 1 - \frac{c(e^{-\alpha x})}{c(\theta)},$$

Which is the exponential power series distributions developed by Chahkandi and Ganjali (2009) that include the lifetime distributions class proposed by Adamidis and Lukas (1998), Kus (2007), Tahmasbi and Rezaei (2008).

(2) Let $Y = G^{-1}(1 - G(X))$, where G^{-1} is the inverse function of G, then Y has a GEWPS distribution as

$$F_Y(y) = P(Y \leq y) = P(G^{-1}(1 - G(X)) \leq y) = P(X \geq G^{-1}(1 - G(y)))$$

$$= 1 - F_{X_{(1)}}(G^{-1}(1 - G(y))) = \frac{c(\theta G(y))}{c(\theta)}.$$

Based on the choice of $a_n, c(\theta), H(x)$ and δ with form (6), this class covers the entire compound truncated discrete distributions with all of the continuous lifetime distributions in the literature.

4. Density, survival, and hazard functions

The probability density functions associated with (6) and (7), respectively, are given by

$$f_{X_{(n)}}(x) = \theta\, g(x) \frac{c'(\theta G(x))}{c(\theta)}$$

$$= \theta\alpha\delta h(x) e^{-\alpha H(x)}(1 - e^{-\alpha H(x)})^{\delta-1} \frac{c'(\theta(1 - e^{-\alpha H(x)})^\delta)}{c(\theta)} \qquad (8)$$

and

$$f_{X_{(1)}}(x) = \theta\, g(x) \frac{c'(\theta(1 - G(x)))}{c(\theta)}$$

$$= \theta\alpha\delta h(x) e^{-\alpha H(x)}(1 - e^{-\alpha H(x)})^{\delta-1} \frac{c'(\theta(1 - (1 - e^{-\alpha H(x)})^\delta))}{c(\theta)}. \qquad (9)$$

The survival functions are given by

$$s_{X_{(n)}}(x) = 1 - \frac{c(\theta G(x))}{c(\theta)} = 1 - \frac{c(\theta(1 - e^{-\alpha H(x)})^\delta)}{c(\theta)} \qquad (10)$$

And

$$s_{X_{(1)}}(x) = \frac{c(\theta(1 - G(x)))}{c(\theta)} = \frac{c(\theta(1 - (1 - e^{-\alpha H(x)})^\delta))}{c(\theta)}. \qquad (11)$$

The corresponding hazard rate functions are

$$\tau_{X_{(n)}}(x) = \frac{f_{X_{(n)}}(x)}{s_{X_{(n)}}(x)} = \theta g(x) \frac{c'(\theta G(x))}{c(\theta) - c(\theta G(x))}$$

$$= \theta\alpha\delta h(x) e^{-\alpha H(x)}(1 - e^{-\alpha H(x)})^{\delta-1} \frac{c'(\theta(1 - e^{-\alpha H(x)})^\delta)}{c(\theta) - c(\theta(1 - e^{-\alpha H(x)})^\delta)} \qquad (12)$$

and

$$\tau_{X_{(1)}}(x) = \frac{f_{X_{(1)}}(x)}{s_{X_{(1)}}(x)} = \theta g(x) \frac{c'(\theta(1 - G(x)))}{c(\theta(1 - G(x)))}$$

$$= \theta\alpha\delta h(x) e^{-\alpha H(x)}(1 - e^{-\alpha H(x)})^{\delta-1} \frac{c'(\theta(1 - (1 - e^{-\alpha H(x)})^\delta))}{c(\theta(1 - (1 - e^{-\alpha H(x)})^\delta))}. \qquad (13)$$

The limiting distribution of the GPS when $\theta \to 0^+$ is

$$\lim_{\theta \to 0^+} F_{X_{(n)}}(x) = \lim_{\theta \to 0^+} \frac{c(\theta G(x))}{c(\theta)} = \lim_{\theta \to 0^+} \frac{\sum_{n=1}^{\infty} a_n \theta^n (G(x))^n}{\sum_{n=1}^{\infty} a_n \theta^n}$$

$$= \lim_{\theta \to 0^+} \frac{a_c (G(x))^c + \sum_{n=c+1}^{\infty} a_n \theta^{n-c} (G(x))^n}{a_c + \sum_{n=c+1}^{\infty} a_n \theta^{n-c}}$$

$$= (G(x))^c = (1 - e^{-\alpha H(x)})^{c\delta}$$

where $c = \min\{n \in \mathbb{N}, a_n > 0\}$.

The densities of GEWPS can be expressed as an infinite number of linear combinations of densities of the order statistics. Given that $c'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1},$ therefore,

$$f_{X_{(n)}}(x) = \theta\, g(x) \frac{c'(\theta G(x))}{c(\theta)} = \sum_{n=1}^{\infty} p(N=n)\, g_{Y_{(n)}}(x;n)$$

where $g_{Y_{(n)}}(x;n)$ is the density function of $Y_{(n)} = \max(Y_1, \ldots, Y_n),$ given by

$$g_{Y_{(n)}}(x;n) = ng(x)(G(x))^{n-1} = n\alpha\delta h(x;\eta) e^{-\alpha H(x)} (1 - e^{-\alpha H(x)})^{n\delta - 1}.$$

Moreover,

$$f_{X_{(1)}}(x) = \theta\, g(x) \frac{c'(\theta - \theta G(x))}{c(\theta)} = \sum_{n=1}^{\infty} p(N=n)\, g_{Y_{(1)}}(x;n)$$

where $g_{Y_{(1)}}(x;n)$ is the density function of $Y_{(1)} = \min(Y_1, \ldots, Y_n),$ given by

$$g_{X_{(1)}}(x;n) = ng(x)(1 - G(x))^{n-1} = n\alpha\delta h(x) e^{-\alpha\delta(n-1)H(x)} (1 - e^{-\alpha H(x)})^{\delta - 1}.$$

5. Quantiles, moments, and order statistics

Let X be a random variable with cdf as in (6). The quantile function, i.e., $Q_{X_n}(p)$, is defined by $F_{X_{(n)}}(Q_{X_{(n)}}(p)) = p, p \in (0,1)$. Therefore,

$$Q_{X_{(n)}}(p) = H^{-1}\left\{ -\frac{1}{\alpha} \log\left[\left(1 - \frac{c^{-1}((pc(\theta))}{\theta}\right)^{\frac{1}{\delta}} \right] \right\},$$

with the cdf as in (7)

$$Q_{X_{(1)}}(p) = H^{-1}\left\{-\frac{1}{\alpha}\log\left[1-\left(1-\frac{c^{-1}((1-p)c(\theta))}{\theta}\right)^{\frac{1}{\delta}}\right]\right\},$$

where $\theta, \alpha > 0, p \in (0,1)$.

The moment generating function is obtained as follows:

$$M_{X_{(n)}}(t) = \int_0^\infty e^{tx} f_{X_{(n)}}(x)dx = \int_0^\infty e^{tx}\sum_{n=1}^\infty P(N=n)g_{X_{(n)}}(x)dx$$

$$= \sum_{n=1}^\infty P(N=n)\int_0^\infty e^{tx} g_{X_{(n)}}(x)dx = \sum_{n=1}^\infty P(N=n)E(Y_{(n)}^k),$$

which can be obtained in the function $H(x)$. Similarly,

$$M_{X_{(1)}}(t) = \int_0^\infty e^{tx} f_{X_{(1)}}(x)dx = \sum_{n=1}^\infty P(N=n)\int_0^\infty e^{tx} g_{X_{(1)}}(x)dx = \sum_{n=1}^\infty P(N=n)E(Y_{(1)}^k).$$

Order statistics isamong the most fundamental tools in non-parametric statistics and inference.It enters estimationproblems and hypothesis tests in many ways.The probability distribution function of the ith order statistics from a random sample $X_1,...,X_m$ withdensity function (8) is given by

$$f^{X_{(n)}}{}_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!}f_{X_{(n)}}(x)\left[\frac{c(\theta G(x))}{c(\theta)}\right]^{i-1}\left[1-\frac{c(\theta G(x))}{c(\theta)}\right]^{m-i}, x > 0.$$

Using the binomial expansion, the above formula can be written as follows:

$$f^{X_{(n)}}{}_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!}f_{X_{(n)}}(x)\sum_{j=0}^{m-i}(-1)^j\binom{m-i}{j}\left[\frac{c(\theta G(x))}{c(\theta)}\right]^{m-j-1}, x > 0.$$

For the density function in (9),

$$f^{X_{(1)}}{}_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!}f_{X_{(1)}}(x)\left[1-\frac{c(\theta G(x))}{c(\theta)}\right]^{i-1}\left[\frac{c(\theta G(x))}{c(\theta)}\right]^{m-i}, x > 0.$$

This expression can be written as

$$f^{X_{(1)}}{}_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!}f_{X_{(1)}}(x)\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\left[\frac{c(\theta G(x))}{c(\theta)}\right]^{m-j-1}, x > 0.$$

6. Estimation and inference

Let $X_1,...,X_n$ be a random sample, with the observed values $x_1,...,x_n$ obtained from the GEWPS with parameters θ, α, δ and ξ. Let $\Theta = (\theta, \alpha, \delta, \xi)$ be the $p \times 1$ parameter vector. The log likelihood function is given by

$$\ln = n[\log\theta + \log\alpha + \log\delta - \log(c(\theta))] - \alpha\sum_{i=1}^{n}H(x_i) + (\delta-1)\sum_{i=1}^{n}\log(1-e^{-\alpha H(x_i)})$$

$$+\sum_{i=1}^{n}\log(c'(\theta(1-e^{-\alpha H(x_i)})^{\delta}))$$

Consider $p_i = 1-e^{-\alpha H(x_i;\eta)}$. Then the score function is given by

$$U_n(\Theta) = (\partial\ln/\partial\theta, \partial\ln/\partial\alpha, \partial\ln/\partial\delta, \partial\ln/\partial\xi)^T$$

.

$$\frac{\partial\ln}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{p_i^{\delta}c''(\theta p_i^{\delta})}{c'(\theta p_i^{\delta})} - n\frac{c'(\theta)}{c(\theta)},$$

$$\frac{\partial\ln}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}H(x_i) + (\delta-1)\sum_{i=1}^{n}\alpha h(x_i)\frac{(1-p_i)}{p_i} + \sum_{i=1}^{n}\frac{\theta\delta H(x_i)(1-p_i)p_i^{\delta-1}c''(\theta p_i^{\delta})}{c'(\theta p_i^{\delta})},$$

$$\frac{\partial\ln}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log(p_i) + \sum_{i=1}^{n}\frac{\theta p_i^{\delta}\log(p_i)(c''(\theta p_i^{\delta}))}{c'(\theta p_i^{\delta})},$$

$$\frac{\partial\ln}{\partial\xi_k} = \sum_{i=1}^{n}\frac{\partial\log h(x_i)}{\partial\xi_k} - \alpha\sum_{i=1}^{n}\frac{\partial H(x_i)}{\partial\xi_k}[1+(\delta-1)\frac{1-p_i}{p_i} + \theta\delta(1-p_i)p_i^{\delta-1}\frac{c''(\theta p_i^{\delta})}{c'(\theta p_i^{\delta})}].$$

The maximum likelihood estimation (MLE) of Θ say $\hat{\Theta}$ is obtained by solving the nonlinear system $U_n(x;\Theta) = 0$. This nonlinear system of equations does not have a closed form. For the interval estimation and hypothesis tests on the model parameters, we require the following observed information matrix:

$$I_n(\Theta) = -\begin{bmatrix} I_{\theta\theta} & I_{\theta\alpha} & I_{\theta\delta} & | & I_{\theta\xi}^T \\ I_{\alpha\theta} & I_{\alpha\alpha} & I_{\alpha\delta} & | & I_{\alpha\xi}^T \\ I_{\delta\theta} & I_{\delta\alpha} & I_{\delta\delta} & | & I_{\delta\xi}^T \\ - & - & - & - & - \\ I_{\theta\xi} & I_{\alpha\xi} & I_{\delta\xi} & | & I_{\xi\xi} \end{bmatrix},$$

where the elements of $I_n(\Theta)$ are the second partial derivatives of $U_n(\Theta)$. Under the standard regular conditions for the large sample approximation (Cox and Hinkley, 1974) fulfilled for the proposed model, the distribution of $\hat{\Theta}$ is approximately $N_p(\Theta, J_n(\Theta)^{-1})$, where $J_n(\Theta) = E[I_n(\Theta)]$. Whenever the parameters are in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is $N_p(0, J(\Theta)^{-1})$, where $J(\Theta)^{-1} = \lim_{n\to\infty}n^{-1}I_n(\Theta)$ is the unit information matrix and p is the number of parameters of the distribution. The asymptotic multivariate normal $N_p(\Theta, I_n(\hat{\Theta})^{-1})$ distribution of $\hat{\Theta}$ can be used to approximate the confidence interval for the parameters, the hazard rate, and the survival functions. An $100(1-\gamma)$ asymptotic confidence interval for parameter Θ_i is given by

$$(\hat{\Theta}_i - Z_{\frac{\gamma}{2}}\sqrt{\hat{I}^{ii}}, \hat{\Theta}_i + Z_{\frac{\gamma}{2}}\sqrt{\hat{I}^{ii}}),$$

Where I^{ii} is the (i,i) diagonal element of $I_n(\Theta)^{-1}$ for $i=1,...,p$ and $Z_{\frac{\gamma}{2}}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

6.1 EM algorithm

Based on the underline distribution, the MLE of the parameters can be found analytically using theEM algorithm. The Newton–Raphson algorithm is one of the standard methods to determine the MLEs of the parameters. To use the algorithm, the second derivatives of the log-likelihood are required for all iterations. The EM algorithm is a very powerful tool forhandling the incomplete data problem (Dempster et al.,1977; McLachlan and Krishnan, 1983). It is an iterative method thatrepeatedly replaces missing data with estimated values and updatesthe parameter estimates. It is especially useful if the complete dataset is easy to analyze. As pointed out by Little and Rubin (1983), the EM algorithm converge reliably but rather slowly, compared withthe Newton–Raphson method, when the amount of information in the missing data is relatively large. Recently, EM algorithm has been used by such researchers asAdamidis and Loukas (1998), Adamidis (1999), Ng et al. (2002), Karlis (2003), and Adamidis et al. (2005).

In estimating Θ, the EM algorithm is a recurrent method in which each step consists of an estimate of the expected value of a hypothetical random variable and later maximizes the log-likelihood of the complete data. Let the complete data be $X_1,...,X_n$, with the observed values $x_1,...,x_n$ and the hypothetical random variable $N_1,...,N_n$. The joint probability function is such that the marginal density of $X_1,...,X_n$ is the likelihood of interest. Then, we define a hypothetical completedata distribution for each $(X_i,N_i)^T, i=1,...,\mathrm{m},$ with a joint probability function in the form of

$$f_{X,N}(\mathrm{x},\mathrm{z};\Theta)=\frac{a_z\theta^z}{c(\theta)}z\,\alpha\delta h(x)e^{-\alpha H(x)}(1-e^{-\alpha H(x)})^{z\delta-1},$$

where $x,\theta,\delta,\alpha \in \mathrm{R}^+$ and $z \in \mathrm{N}$. Therefore, it is straightforward to verify that the E-step of an EM cycle requires the computation of the conditional expectation of $(Z|X;\Theta^{(r)})$, where $\Theta^{(r)}=(\theta^{(r)},\alpha^{(r)},\delta^{(r)},\xi^{(r)})$ is the current estimate (in the rth iteration) of Θ. Then, the EM cycle is completed with the M-step, which is complete data maximum likelihood over (Θ), with the missing Z's replaced by their conditional expectations $E(Z|X;\Theta)$ (Adamidis and Loukas,1998), where

$$p_{Z|X}(z)=\frac{f_{X,N}(x,z;\Theta)}{f_X(x)}=\frac{a_z\theta^{z-1}z\,(1-e^{-\alpha H(x)})^{z\delta-\delta}}{c'(\theta(1-e^{-\alpha H(x)})^\delta)},$$

$$c'(\theta)+c''(\theta)=\sum_{z=1}^{\infty}z^2a_z\theta^{z-1}$$
and since   its expected value is

$$E(Z|X)=\sum_{z=1}^{\infty}\frac{a_z\theta^{z-1}z^2(1-e^{-\alpha H(x)})^{z\delta-\delta}}{c'(\theta(1-e^{-\alpha H(x)})^\delta)}=\frac{1}{c'(\theta(1-e^{-\alpha H(x)})^\delta)}\sum_{z=1}^{\infty}z^2a_z[\theta(1-e^{-\alpha H(x)})^\delta]^{z-1}$$

$$=\frac{c'(\theta(1-e^{-\alpha H(x)})^\delta)+\theta(1-e^{-\alpha H(x)})^\delta c''(\theta(1-e^{-\alpha H(x)})^\delta)}{c'(\theta(1-e^{-\alpha H(x)})^\delta)}$$

$$=1+\frac{\theta(1-e^{-\alpha H(x)})^\delta c''(\theta(1-e^{-\alpha H(x)})^\delta)}{c'(\theta(1-e^{-\alpha H(x)})^\delta)}.$$

7. **Special subclasses**

In this section, we present four special subclasses of the GEWPS family of distributions. We provide the forms of the cumulative, density survival, and hazard rate functions for $X_{(1)}$ and $X_{(n)}$.

6.1 A compound class of the Poisson and lifetime distributions

The compound class of the Poisson distribution (CP) (Alkarni and Oraby, 2012) is a subclass of the GEWPS family of distribution, with $a_n = \dfrac{1}{n!}, c(\theta) = e^{\theta} - 1, (\theta > 0)$. We assume that $X_1, ..., X_N$ are identically independent random variables with a distribution function as in (3) and with N followinga truncated Poisson distribution at zero. Table 2 shows the necessary functions for this class.

Table 2: Cdf, pdf, survival, and hazard rate functions for the CL class

$X_{(1)} = \min(X_1, ..., X_N)$	$X_{(n)} = \max(X_1, ..., X_N)$
$F(x) = \dfrac{1 - e^{-\theta G(x)}}{1 - e^{-\theta}}$	$F(x) = \dfrac{e^{-\theta(1-G(x))} - e^{-\theta}}{1 - e^{-\theta}}$
$f(x) = \dfrac{\theta g(x) e^{-\theta G(x)}}{1 - e^{-\theta}}$	$f(x) = \dfrac{\theta g(x) e^{-\theta(1-G(x))}}{1 - e^{-\theta}}$
$s(x) = \dfrac{e^{-\theta G(x)} - e^{-\theta}}{1 - e^{-\theta}}$	$s(x) = \dfrac{1 - e^{e^{-\theta(1-G(x))}}}{1 - e^{-\theta}}$
$\tau(x) = \dfrac{\theta g(x) e^{-\theta G(x)}}{e^{-\theta G(x)} - e^{-\theta}}$	$\tau(x) = \dfrac{\theta g(x) e^{e^{-\theta(1-G(x))}}}{1 - e^{e^{-\theta(1-G(x))}}}$

7.2 A compound class of logarithmic and lifetimes distributions

The compound class of logarithmic distribution (CL) (Alkarni, 2012) is a subclass of the GEWPS family of distribution, with $a_n = 1, c(\theta) = -\log(1-\theta), \theta \in (0,1)$. We assume that $X_1, ..., X_N$ are identically independent random variables with a distribution function as in (3) and with N followinga truncated logarithmic distribution at zero. Table 3 shows the necessary functions for this class.

Table 3: Cdf, pdf, survival, and hazard rate functions for the CL class

$X_{(1)} = \min(X_1, ..., X_N)$	$X_{(n)} = \max(X_1, ..., X_N)$
$F(x) = 1 - \dfrac{\log(1 - \theta(1 - G(x)))}{\log(1 - \theta)}$	$F(x) = \dfrac{\log(1 - \theta G(x))}{\log(1 - \theta)}$
$f(x) = \dfrac{-\theta g(x)}{(1 - \theta(1 - G(x)))\log(1 - \theta)}$	$f(x) = \dfrac{-\theta g(x)}{(1 - \theta G(x))\log(1 - \theta)}$
$s(x) = \dfrac{\log(1 - \theta(1 - G(x)))}{\log(1 - \theta)}$	$s(x) = 1 - \dfrac{\log(1 - \theta G(x))}{\log(1 - \theta)}$
$\tau(x) = \dfrac{-\theta g(x)}{\log(1 - \theta(1 - G(x)))[1 - \theta(1 - }$	$\tau(x) = \dfrac{\theta g(x)}{(1 - \theta G(x))[\log(1 - \theta G(x)) - \log}$

7.3 A compound class of geometric and lifetime distributions

The compound class of geometric distribution (CG) (Alkarni, 2013) is a subclass of the GEWPS family of distribution, with $a_n = 1, c(\theta) = \theta(1 - \theta)^{-1}, \theta \in (0,1)$.

We assume that $X_1, ..., X_N$ are identically independent random variables with adistribution function as in (3) and with N followinga truncated geometric distribution at zero. Table 4 showsthe necessary functions for this class.

Table 4: Cdf, pdf, survival, and hazard rate functions for the CG class

$X_{(1)} = \min(X_1, ..., X_N)$	$X_{(n)} = \max(X_1, ..., X_N)$
$F(x) = 1 - \dfrac{(1-\theta)(1-G(x))}{1-\theta(1-G(x))}$	$F(x) = \dfrac{(1-\theta)G(x)}{1-\theta G(x)}$
$f(x) = \dfrac{(1-\theta)g(x)}{(1-\theta(1-G(x)))^2}$	$f(x) = \dfrac{(1-\theta)g(x)}{(1-\theta G(x))^2}$
$s(x) = \dfrac{(1-\theta)(1-G(x))}{1-\theta(1-G(x))}$	$s(x) = 1 - \dfrac{(1-\theta)G(x)}{1-\theta G(x)}$
$\tau(x) = \dfrac{g(x)}{(1-\theta(1-G(x)))(1-G(}$	$\tau(x) = \dfrac{(1-\theta)g(x)}{(1-\theta G(x))(1-G(x}$

7.4 A compound class of binomial and lifetime distributions

The compound class of binomial distribution (CB) (Alkarni, 2013) is a subclass of the GEWPS family of distributions with $c(\theta) = (\theta+1)^m - 1$. We assume that $X_1, ..., X_N$ are identically independent random variables with a distributionfunction as in (3) and with N following a truncated binomial distribution at zero. Table 5 shows the necessary functions for this class.

Table 5: Cdf, pdf, survival, and hazard rate functions for the CB class

$X_{(1)} = \min(X_1, ..., X_N)$	$X_{(n)} = \max(X_1, ..., X_N)$
$F(x) = 1 - \dfrac{(\theta(1-G(x))+1)^m}{(\theta+1)^m - 1}$	$F(x) = \dfrac{(\theta G(x)+1)^m - 1}{(\theta+1)^m - 1}$
$f(x) = \dfrac{m\theta g(x)(\theta(1-G(x))}{(\theta+1)^m - 1}$	$f(x) = \dfrac{m\theta g(x)(\theta G(x)+}{(\theta+1)^m - 1}$
$s(x) = \dfrac{(\theta(1-G(x))+1)^m - 1}{(\theta+1)^m - 1}$	$s(x) = 1 - \dfrac{(\theta G(x)+1)^m -}{(\theta+1)^m - 1}$
$\tau(x) = \dfrac{m\theta g(x)(\theta(1-G(x))+}{(\theta(1-G(x))+1)^m}$	$\tau(x) = \dfrac{m\theta g(x)(\theta G(x)+}{(\theta+1)^m - (\theta G(x)}$

Table 6 illustrates examples of some existing distributions with $H(x)$ and $c(\theta)$. The other functions can be obtained directly from the previous tables.

Table 6: Special distributions with cdf and the corresponding $H(x)$ and $c(\theta)$

Distribution	$H(x)$	δ	a_n	$c(\theta)$	$F_{X_{(n)}}(x)$	References
Generalized geometric exponential	x	δ	1	$\dfrac{\theta}{1-\theta}$	$\dfrac{(1-\theta)(1-e^{-\alpha x})^{\delta}}{1-\theta((1-e^{-\alpha x})^{\delta}}$	Mahmoudi and Jafari (2012)
Exponentiated Weibull-geometric	$(\beta x)^{\gamma}$	δ	1	$\dfrac{\theta}{1-\theta}$	$\dfrac{(1-\theta)(1-e^{-(\beta x)^{\gamma}})^{\delta}}{1-\theta((1-e^{-(\beta x)^{\gamma}})^{\delta}}$	Mahmoudi and Shiran(2012)
Generalized Poisson exponential	x	δ	$\dfrac{1}{n!}$	$e^{\theta}-1$	$\dfrac{e^{\theta(1-e^{-\alpha x})^{\delta}}-1}{e^{\theta}-1}$	Mahmoudi and Jafari (2012)
Exponentiated Weibull Poisson	$(\beta x)^{\gamma}$	δ	$\dfrac{1}{n!}$	$e^{\theta}-1$	$\dfrac{e^{\theta(1-e^{-(\beta x)^{\gamma}})^{\delta}}-1}{e^{\theta}-1}$	Mahmoudi and Sepahdar (2013)
Generalized binomial exponential	x	δ	$\dbinom{m}{n}$	$(\theta+1)^{m}-1$	$\dfrac{(\theta(1-e^{-\alpha x})^{\delta}+1)^{m}-1}{(\theta+1)^{m}-1}$	Mahmoudi and Jafari (2012)
Generalized logarithmic exponential	x	δ	$\dfrac{1}{n}$	$-\log(1-\theta)$	$\dfrac{\log(1-\theta(1-e^{-\alpha x})^{\delta})}{\log(1-\theta)}$	Mahmoudi and Jafari (2012)
Exponentiated Weibull-logarithmic	$(\beta x)^{\gamma}$	δ	$\dfrac{1}{n}$	$-\log(1-\theta)$	$\dfrac{\log(1-\theta(1-e^{-(\beta x)^{\gamma}})^{\delta})}{\log(1-\theta)}$	Mahmoudi and Sepahdar (2014)
Pareto Poisson	$\log(x/k)$	1	1	$e^{\theta}-1$	$1-\dfrac{e^{\theta(k/x)^{\alpha}}-1}{e^{\theta}-1};\ x \geq k$	Silva et al. (2013)
Poisson-Lomax	$(1+\beta x)^{-\alpha}$	1	1	$e^{\theta}-1$	$\dfrac{e^{\theta(1+\beta x)^{-\alpha}}-1}{e^{\theta}-1}$	Al-Zahrani and Sagor (2014)

8. Submodels and applications

In this section, two models are discussed with real data as examples of the GEWPS family. Geometric exponential distribution (GE) and generalized exponentialgeometric (GEG) distribution are fitted for real data.By substitutingdirectly in the forms found in Table 4 from Section 6 for $X_{(n)} = \max\{X_i\}_{i=1}^{N}$, we obtain the following pdfs and hazard functions:

$$f_{GE}(x;\theta,\beta) = \frac{(1-\theta)\beta e^{-\beta x}}{[1-\theta(1-e^{-\beta x})]^2},$$

$$\tau_{GE}(x;\theta,\beta) = \frac{\beta(1-\theta)}{1-\theta(1-e^{-\beta x})}$$

$$f_{GEG}(x;\theta,\beta,\delta) = \frac{(1-\theta)\beta\delta e^{-\beta x}(1-e^{-\beta x})^{\delta-1}}{[1-\theta(1-e^{-\beta x})^{\delta}]^2},$$

$$\tau_{GEG}(x;\theta,\beta,\delta) = \frac{(1-\theta)\beta\delta e^{-\beta x}(1-e^{-\beta x})^{\delta-1}}{(1-\theta(1-e^{-\beta x})^{\delta})(1-(1-e^{-\beta x})^{\delta})}.$$

Figs. 1 and 2 show the densities and hazard functions of the GEand GEGdistributions for the selected parameter values.
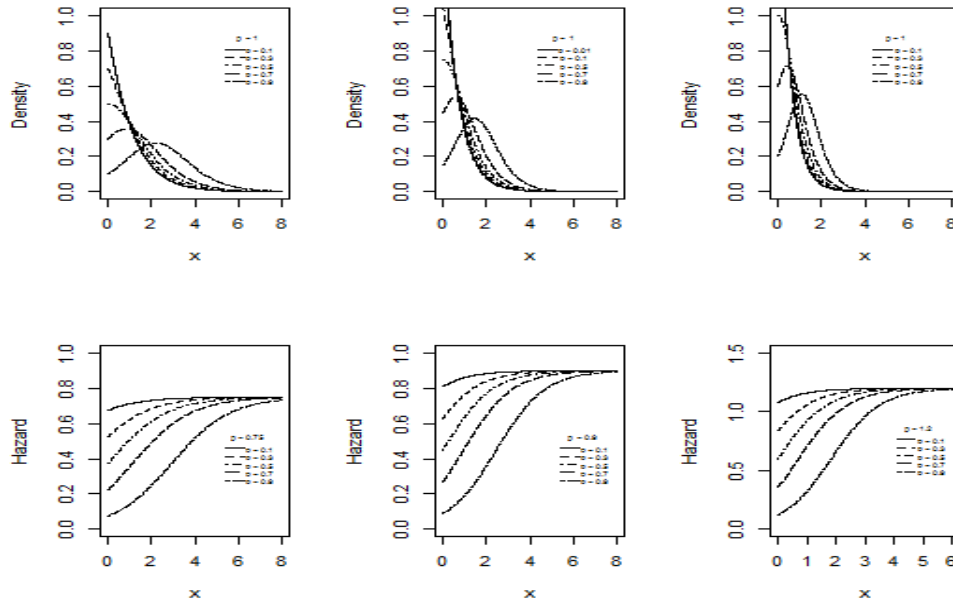

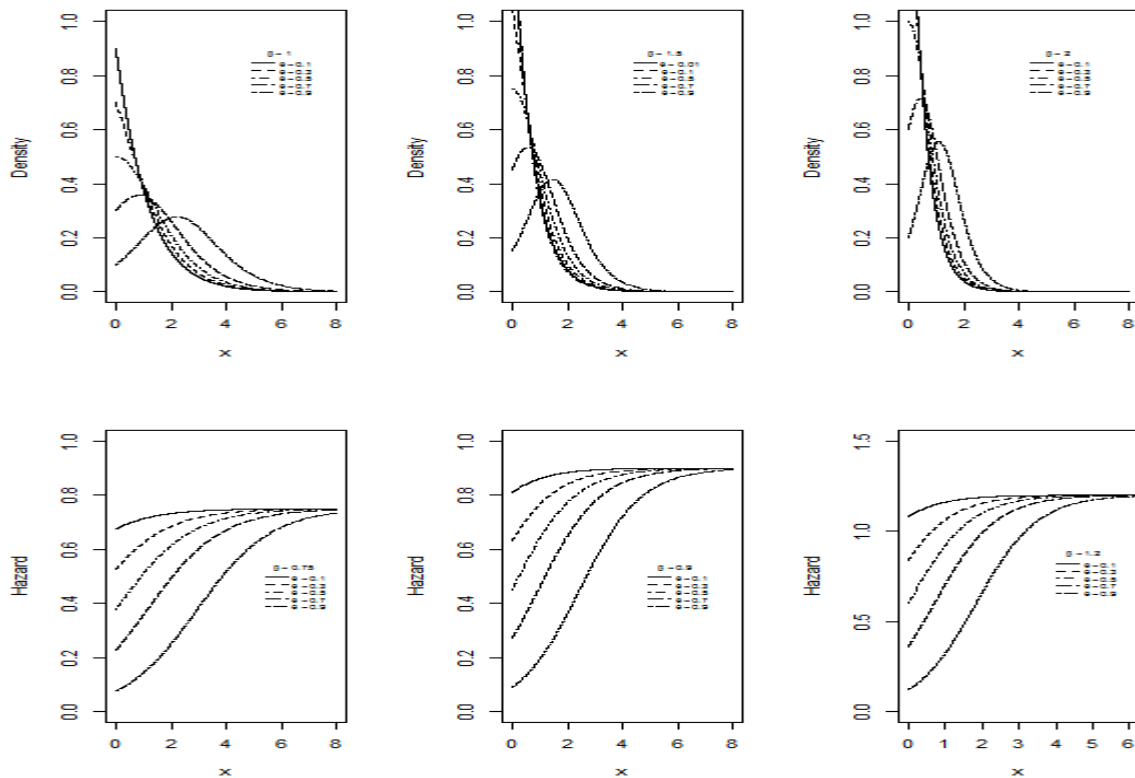**Fig. 1. Plots of the density and hazard rate function of the GE for different values of β and θ .**

Fig. 2. Plots of the density and hazard rate function of the GEG for $\beta = 1$ and different values δ and θ .

Both models are fitted for the data introduced by Birnbaum and Saunders (1969) on the fatigue life of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second. The data arelisted in Table 3, which consists of 101 observations.

Table 7: Fatigue life of 6061-T6 aluminum coupons

70   90   96 97 99 100 103 104 104 105 107 108 108 108 109
109 112 112 113 114 114 114 116 119 120 120 120 121 121 123
124 124 124 124 124 128 128 129 129 130 130 130 131 131 131
131 131 132 132 132 133 134 134 134 134 134 136 136 137 138
138 138 139 139 141 141 142 142 142 142 142 142 144 144 145
146 148 148 149 151 151 152 155 156 157 157 157 157 158 159
162 163 163 164 166 166 168 170 174 196 212

The EMalgorithm is used to estimate the model parameters. The MLEs of the parameters, the maximized log likelihood, the Kolmogorov–Smirnov statistics with its respective p-value, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for the GE and GEG models are given in Table 8. The fitted densities and the empirical distribution versus the fitted cdfs of the GE and GEG models ofthis data are shown in Fig. 4. They indicate that the GEG distribution fits the data better than the GE distribution. The KS test statistic takes the smallest value with the largest value of its corresponding p-value for the GEG distribution. Moreover, this conclusion is confirmed from the AIC and BIC values for the fitted models given in Table 8.Their densities and cumulative distributions are plotted in Fig.4.

Table 8: Parameter estimates, KS statistic, P-value, AIC, and BIC of the Birnbaum and Saunders data.

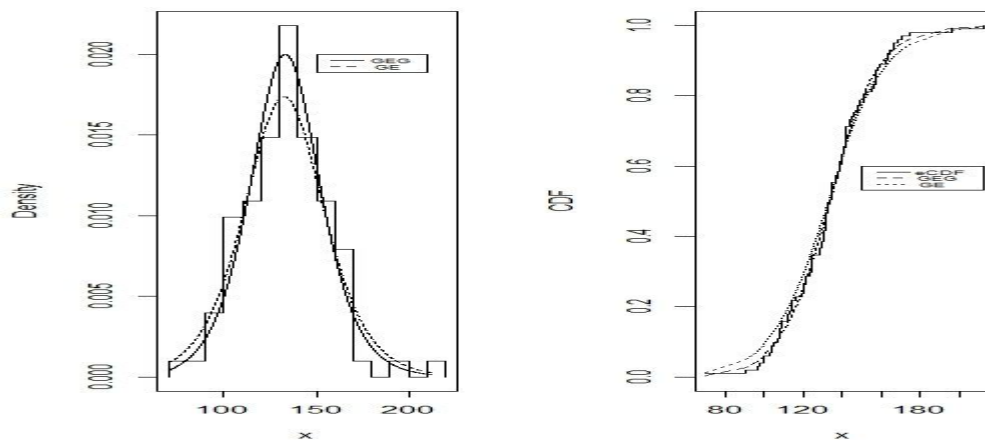| Dist.	MLE(std.)	K-S	p-value	-log(L)	AIC	BIC
GE	$\hat{\theta} = 0.9999$ $\hat{\beta} = 0.0695$	0.076	0.6041	457.14	918.28	923.51
GEG	$\hat{\theta} = 0.9884$ $\hat{\beta} = 0.0795$ $\hat{\delta} = 463.5169$	0.0492	0.9673	455.2	916.4	924.245

Fig. 3: Plots of fitted GEG and GE of the Birnbaum and Saunders data.

9. **Concluding remarks**

We define a new family of lifetime distributions called the GEWPS family of distributions, which generalizes the extended Weibull power series class and the generalized power series exponential distributions introduced by Silva et al. (2013) andMahmoudi and Jafari (2012), respectively. The GEWPS class contains many lifetime subclasses and distributions. Various standard mathematical properties were derived, such as density andsurvival, and hazard functions were introduced in flexible and usefulforms. Parameter estimation using theEMalgorithm was conducted using the maximum likelihood method. Finally, we fitted some of the GEWPS models to a real dataset to show the flexibility and the benefits of the proposed class.

Acknowledgments

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