American Review of Mathematics and Statistics
December 2015, Vol. 3, No.2, pp. 15-23
ISSN: 2374-2348 (Print), 2374-2356 (Online)

Published by American Research Institute for Policy Development
DOI: 10.15640/arms.v3n2a3
URL: http://dx.doi.org/10.15640/arms.v3n2a3

# Nonlinear Retarded Integral Inequalities of Gronwall Type

**Zareen .A.Khan**[1]

**Abstract**

In this paper we establish some new nonlinear integral inequalities of Gronwall-Bellman type. These inequalities generalize some famous inequalities which provide explicit bounds on unknown functions. The inequalities given here can be used as handy tools to study the qualitative as well as quantitative properties of solutions of some nonlinear ordinary differential and integral equations.

**Keyword**: Integral inequalities, nondecreasing, integral equations, ordinary differential equations.

## 1. Introduction

The integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations.

**Lemma1.1.**Gronwall in 1919 established the following inequality:

Let $x(t), a(t)$ and $f(t)$ be real-valued nonnegative continuous functions defined on $I=[0,\infty)$ with $f(t)\geq 0$ .If the inequality

$$x(t)\leq a(t)+\int_0^t f(s)x(s)ds\,,\quad t\in I$$

holds for all $t\in I$ ,then

$$x(t)\leq a(t)+\int_0^t f(s)a(s)\exp\left(\int_0^s f(\lambda)d\lambda\right),\qquad t\in I \tag{1.1}$$

**Lemma1.2:** Bellman in 1943 studied the following inequality:

Let $x(t)$ and $f(t)$ be real-valued nonnegative continuous functions defined on $I=[0,\infty)$ and $x_0$ is a constant.If the inequality

$$x(t)\leq x_0+\int_0^t f(s)x(s)ds\,,\quad t\in I$$

holds for all $t\in I$ ,then

$$x(t)\leq x_0\exp\left(\int_0^t f(s)ds\right),\qquad t\in I \tag{1.2}$$

**Lemma1.3:** Lipovan in 2000 established the following inequality:

[1]Department of Mathematics, Princess NouraBint Abdul-Rehman University, Riyadh,KSA, dr.zareenkhan@ymail.com

Let $x,\ f \in C([t_0.T_0],\Re_+)$ .Further let $\alpha \in C([t_0.T_0],[t_0,T_0])$ be nondecreasing with $\alpha(t) \in t$ on $[t_0,T_0]$ , and let $k$ be a nonnegative constant.If the inequality

$$x(t) \le k + \int_{\alpha(t_0)}^{\alpha(t)} f(s)x(s)ds, \quad t_0 < t < T_0$$

Implies that

$$x(t) \le k \exp\left( \int_{\alpha(t_0)}^{\alpha(t)} f(s)ds \right), \quad t_0 < t < T_0 \tag{1.3}$$

Many of the results of Gronwall-Bellman can be cited in [1-11].

**2. Main results:** In this section,some new retarded integral inequalities of Gronwall-Bellmantype are introduced. Throughout this paper, $\Re$ denotes the set of real numbers, $I = [0,\infty)$, $\Re_+^* = (0,\infty)$, $\Re_1 = [1,\infty)$. $C(I,I)$ Denotes the set of all nonnegative real-valued continuous functions from $I$ into $I$ and $C^1(I,I)$ denotes the set of all nonnegative real-valued continuously differentiable functions from $I$ into $I$ .

**Theorem 2.1:** Let $x(t),\ f(t)$ and $g(t) \in C(I,\Re_+^*)$ , $\alpha \in C^1(I,I)$ be nondecreasing with $\alpha(t) \le t$ on $I$ .If the inequality

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)\left[ x^{(2-p)}(s) + \int_0^s g(\lambda)x^q(\lambda)d\lambda \right]^p ds, \tag{2.1}$$

holds for all $t \in I$ where $x_0 > 0,\ 0 < p \le 2, 0 \le q < 1$, are constants. Then

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))ds, \quad \forall t \in I \tag{2.2}$$

Where
$$k_1(t) = \exp\left[ p(2-p)\int_0^{\alpha(t)} f(s)ds \right]\left[ x_0^{(1-q)(2-p)} + (1-q)\int_0^{\alpha(t)} g(s)\exp\left[ -(2-p)(1-q)\int_0^s f(\lambda)d\lambda \right]ds \right]^{\frac{p}{1-q}}, \tag{2.3}$$

for all $t \in I$ .

**Proof:** Let $M(t)$ be defined as a function by the right-hand side of (2.1).Then

$$x(t) \le M(t), \quad \text{or} \quad x(\alpha(t)) \le M(\alpha(t)) \le M(t) \ \forall t \in I \tag{2.4}$$

Differentiating $M(t)$ with respect to $t$ and using (2.4) implies that

$$\frac{dM(t)}{dt} \le \alpha'(t) f(\alpha(t)) L^p(t) \tag{2.5}$$

where $L(t)=M^{(2-p)}(t)+\int_0^{\alpha(t)} g(s)M^{q}(s)ds,$ thus we have $L(0)=M^{(2-p)}(0)=x_0^{(2-p)}$, and

$$M(t)\le L(t), \quad \forall t\in I \tag{2.6}$$

Differentiating $L(t)$ with respect to $t$ and using (2.5) and (2.6) leads to

$$\frac{dL(t)}{dt}\le (2-p)\alpha'(t)f(\alpha(t))L(t)+\alpha'(t)g(\alpha(t))L^{q}(t) \tag{2.7}$$

Since $L(t)>0$, then the inequality (2.7) can be rewritten as

$$L^{-q}(t)\frac{dL(t)}{dt}-(2-p)\alpha'(t)f(\alpha(t))L^{(1-q)}(t)\le \alpha'(t)g(\alpha(t)) \tag{2.8}$$

By substituting

$$z(t)=L^{(1-q)}(t), \tag{2.9}$$

We have

$$z(0)=L^{(1-q)}(0)\le x_0^{(1-q)(2-p)} \tag{2.10}$$

and $L^{-q}(t)\dfrac{dL(t)}{dt}=\dfrac{1}{(1-q)}\dfrac{dz(t)}{dt}$

then inequality (2.8) takes the form

$$\frac{dz(t)}{dt}-(1-q)(2-p)\alpha'(t)f\alpha(t))z(t)\le (1-q)\alpha'(t)g(\alpha(t)) \tag{2.11}$$

The inequality (2.11) implies the estimation for $z(t)$ by using (2.10) as

$$z(t)\le \exp\left[(1-q)(2-p)\int_0^{\alpha(t)} f(s)ds\right]\left[x_0^{(1-q)(2-p)}+(1-q)\int_0^{\alpha(t)} g(s)\exp\left[-(2-p)(1-q)\int_0^{s} f(\lambda)d\lambda\right]ds\right]$$

$\forall t\in I$ .By using (2.9) ,the above inequality takes the form

$$L^{p}(t)\le k_1(t) \tag{2.12}$$

where $k_1(t)$ is defined as in (2.3).By substituting (2.12) in (2.5) we observe that

$$\frac{dM(t)}{dt}\le \alpha'(t)f(\alpha(t))k_1(t) \tag{2.13}$$

By integrating both sides of inequality (2.13) from $0$ to $\alpha(t)$ and using (2.6) yields

$$M(t)\le x_0+\int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))ds, \quad \forall t\in I \tag{2.14}$$

Using (2.14) in (2.4), we get the inequality (2.2).

**Remark:** If we put $g(t)=0$ , $p=1$, $x_0=a(t)$, $q=1$ and $\alpha(t)=t$ in Theorem2.1 then it reduces to Lemma1.1.

**Theorem 2.2:** Let $x(t)$, $f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \le t$ on $I$. If the inequality

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)\left[x^p(s) + \int_0^s g(\lambda)x^{(2p-1)}(\lambda)d\lambda\right]^p ds, \tag{2.15}$$

holds for all $t \in I$ where $x_0 > 0,\ p \in (0,1)$ are constants. Then

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))ds, \quad \forall t \in I \tag{2.16}$$

where $k_2(t) = \left[x_0^{2p(1-p)} + 2(1-p)\left[p\int_0^{\alpha(t)} f(s)ds + \int_0^{\alpha(t)} g(s)ds\right]\right]^{\frac{p}{2(1-p)}}$, (2.17)

for all $t \in I$.

**Proof:** Let $M(t)$ be defined as a function by the right-hand side of (2.15).Then

$$x(t) \le M(t), \quad \text{or} \quad x(\alpha(t)) \le M(\alpha(t)) \le M(t) \ \forall t \in I \tag{2.18}$$

Differentiating $M(t)$ with respect to $t$ and using (2.18) implies that

$$\frac{dM(t)}{dt} \le \alpha'(t) f(\alpha(t))L^p(t) \tag{2.19}$$

where $L(t) = M^p(t) + \int_0^{\alpha(t)} g(s)M^{(2p-1)}(s)ds,$ thus we have $L(0) = M^p(0) = x_0^p$, and

$$M(t) \le L(t), \quad \forall t \in I \tag{2.20}$$

Differentiating $L(t)$ with respect to $t$ and using (2.19) and (2.20) leads to

$$\frac{dL(t)}{dt} \le \left[p\alpha'(t) f(\alpha(t)) + \alpha'(t)g(\alpha(t))\right]L^{(2p-1)}(t)$$

which can be rewritten as

$$L^{(1-2p)}(t)\frac{dL(t)}{dt} \le \left[p\alpha'(t) f(\alpha(t)) + \alpha'(t)g(\alpha(t))\right] \tag{2.21}$$

By integrating both sides of inequality (2.21) from 0 to $\alpha(t)$ and using (2.20) yields

$$L^p(t) \le k_2(t) \tag{2.22}$$

where $k_2(t)$ is defined as in (2.17).By substituting (2.22) in (2.19) we observe that

$$\frac{dM(t)}{dt} \le \alpha'(t) f(\alpha(t))k_2(t) \tag{2.23}$$

By integrating both sides of inequality (2.23) from 0 to $\alpha(t)$ and using (2.20) yields

$$M(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))ds, \quad \forall t \in I \tag{2.24}$$

Using (2.24) in (2.18), we get the inequality (2.16).

**Theorem 2.3:**Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I,I)$ be nondecreasing with $\alpha(t) \le t$ on $I$ .If the inequality

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)\left[ x^p(s) + \int_0^s g(\lambda)x^p(\lambda)d\lambda \right] ds, \tag{2.25}$$

holds for all $t \in I$ where $x_0 > 0,\ 0 < p < 1$ are constants. Then

where $$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)\left[ x_0^{p(1-p)} + (1-p)\left[ p\int_0^s f(\lambda)d\lambda + \int_0^s g(\lambda)d\lambda \right]\right]^{\frac{1}{(1-p)}} \tag{2.26}$$

for all $t \in I$ .

**Proof:** Let $M(t)$ be defined as a function by the right-hand side of (2.25).Then

$$x(t) \le M(t), \quad \text{or} \quad x(\alpha(t)) \le M(\alpha(t)) \le M(t)\ \forall t \in I \tag{2.27}$$

Differentiating $M(t)$ with respect to $t$ and using (2.27) implies that

$$\frac{dM(t)}{dt} \le \alpha'(t) f(\alpha(t)) L(t) \tag{2.28}$$

where $\quad L(t) = M^p(t) + \int_0^{\alpha(t)} g(s)M^p(s)ds,$ thus we have $L(0) = M^p(0) = x_0^p$, and

$$M(t) \le L(t), \quad \forall t \in I \tag{2.29}$$

Differentiating $L(t)$ with respect to $t$ and using (2.28) and (2.29) leads to

$$\frac{dL(t)}{dt} \le \left[ p\alpha'(t) f(\alpha(t)) + \alpha'(t)g(\alpha(t)) \right] L^p(t)$$

which can be rewritten as

$$L^{-p}(t)\frac{dL(t)}{dt} \le \left[ p\alpha'(t) f(\alpha(t)) + \alpha'(t)g(\alpha(t)) \right] \tag{2.30}$$

By integrating both sides of inequality (2.30) from 0 to $\alpha(t)$ and using (2.29) yields

$$L(t) \le \left[ x_0^{p(1-p)} + (1-p)\left[ p\int_0^s f(\lambda)d\lambda + \int_0^s g(\lambda)d\lambda \right]\right]^{\frac{1}{(1-p)}} \tag{2.31}$$

By substituting (2.31) in (2.29) and using the fact that $M(0) = x_0$ and by integrating both sides of resulting inequality from 0 to $\alpha(t)$ and using (2.27) also we get the required inequality (2.26).

**Theorem 2.4:**Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I,I)$ be nondecreasing with $\alpha(t) \le t$ on $I$ .If the inequality

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)\left[x^{(2-p)}(s) + \int_0^s g(\lambda)x(\lambda)d\lambda\right]^p ds, \quad (2.32)$$

holds for all $t \in I$ where $x_0 > 0,\ 0 < p \le 2$, are constants. Then

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_3(\alpha^{-1}(s))ds, \quad \forall t \in I \quad (2.33)$$

where $k_3(t) = x_0^{p(2-p)}\left[\exp\left[(2-p)\int_0^{\alpha(t)} f(s)ds + \int_0^{\alpha(t)} g(s)ds\right]\right]^p$, (2.34)

for all $t \in I$.

**Proof:** The proof of Theorem2.4 is the same as the proof of Theorem2.1 with suitable modifications.

**Theorem 2.5:**Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \le t$ on $I$ and let $n(t) \in C(I, I)$ be nondecreasing.If the inequality

$$x(t) \le n(t) + \int_0^{\alpha(t)} f(s)x^p(s)ds + \int_0^{\alpha(t)} g(s)x(s)ds, \quad (2.35)$$

holds for all $t \in I$ where $p < 1$, are constants. Then

$$x(t) \le n(t)k_4(t), \quad \forall t \in I \quad (2.36)$$

where $k_4(t) = \exp\left[\int_0^{\alpha(t)} g(s)ds\right]\left[1 + (1-p)\int_0^{\alpha(t)} f(s)n^{-(1-p)}(s)\exp\left[-(1-p)\int_0^s g(\lambda)d\lambda\right]ds\right]^{\frac{1}{1-p}}$, (2.37)

for all $t \in I$.

**Proof:** Since $n(t)$ is positive,monotonicnondecreasing function then inequality (2.35) can be written as

$$\left[\frac{x(t)}{n(t)}\right] \le 1 + \int_0^{\alpha(t)} f(s)n^{-(1-p)}(s)\left[\frac{x(s)}{n(s)}\right]^p ds + \int_0^{\alpha(t)} g(s)\left[\frac{x(s)}{n(s)}\right]ds, \quad (2.38)$$

Let $M(t)$ be defined as a function by the right-hand side of (2.38).Then

$$\frac{x(t)}{n(t)} \le M(t), \quad \text{or} \quad \frac{x(\alpha(t))}{n(\alpha(t))} \le M(\alpha(t)) \le M(t) \ \forall t \in I \quad (2.39)$$

and $M(0) \le 1$ (2.40)

Differentiating $M(t)$ with respect to $t$ and using (2.39) implies that

$$\frac{dM(t)}{dt} \le \alpha'(t)f(\alpha(t))n^{-(1-p)}(\alpha(t))M^p(t) + \alpha'(t)g(\alpha(t))M(t) \quad (2.41)$$

Since $M(t) > 0$ ,then the inequality(2.41) takes the form

$$M^{-p}(t)\frac{dM(t)}{dt} - \alpha'(t)g(\alpha(t))M^{(1-p)}(t) \le \alpha'(t)f(\alpha(t))n^{-(1-p)}(\alpha(t)) \quad (2.42)$$

By substituting $$z(t) = M^{(1-p)}(t)\ ,\qquad (2.43)$$

we have $$z(0) = M^{(1-p)}(0) \le 1 \qquad (2.44)$$

and $M^{-p}(t)\dfrac{dM(t)}{dt} = \dfrac{1}{(1-p)}\dfrac{dz(t)}{dt}$ then inequality (2.42) yields

$$\frac{dz(t)}{dt} - (1-p)\alpha'(t)g\alpha(t))z(t) \le (1-p)\alpha'(t)f(\alpha(t))n^{-(1-p)}(\alpha(t)) \qquad (2.45)$$

The inequality (2.45) implies the estimation for $z(t)$ by using (2.44) as

$$z(t) \le \exp\left[(1-p)\int_0^{\alpha(t)} g(s)ds\right]\left[1+(1-p)\int_0^{\alpha(t)} f(s)n^{-(1-p)}(s)\exp\left[-(1-p)\int_0^s g(\lambda)d\lambda\right]ds\right]^{\frac{1}{1-p}}$$

$\forall t \in I$ .By using (2.43) ,the above inequality takes the form

$$M(t) \le k_4(t) \qquad (2.46)$$

where $k_4(t)$ is defined as in (2.37).By substituting (2.46) in (2.39) we observe that

$$x(t) \le n(t)k_4(t)$$

**Remark:** If we put $f(t) = 0$ , $p = 0$, $n(t) = x_0$ and $\alpha(t) = t$ in Theorem2.5 then it reduces to Lemma1.2.

**Theorem 2.6:**Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \le t$ on $I$ and let $n(t) \in C(I, I)$ be nondecreasing.If the inequality

$$x^p(t) \le n^p(t) + \int_0^{\alpha(t)} f(s)x^p(s)ds + \int_0^{\alpha(t)} g(s)x(s)ds\,, \qquad (2.47)$$

holds for all $t \in I$ where $p > 1$ is constants. Then

$$x(t) \le n(t)k_5(t), \quad \forall t \in I \qquad (2.48)$$

where $$k_5(t) = \exp\left[\frac{1}{p}\int_0^{\alpha(t)} f(s)ds\right]\left[1+\frac{(p-1)}{p}\int_0^{\alpha(t)} g(s)n^{-(p-1)}(s)\exp\left[-\frac{(p-1)}{p}\int_0^s f(\lambda)d\lambda\right]ds\right]^{\frac{1}{p-1}}, \qquad (2.49)$$

for all $t \in I$.

**Proof:** The proof of Theorem2.6 is the same as the proof of Theorem2.5 with suitable modifications.

**3. Application:**In this section we present an application of the inequality given in Theorem2.3 to illustrate the usefulness of our results.

Consider the retarded integral equation

$$\begin{cases} \frac{dx(t)}{dt} = M(t, x(\alpha(t)), \int_0^t H(s, x(\alpha(s)))ds, \quad \forall t \in I \\ x(0) = x_0 \end{cases} \tag{3.1}$$

where $M \in C(I \times \Re^3, \Re), H \in C(I \times \Re, \Re)$, $|x_0| > 0$ is a constant,satisfy the following conditions

$$|H(t, x(t)| \le g(t)|x(t)|^p \tag{3.2}$$

$$\left| M(t, x(\alpha(t)), \int_0^t H(s, x(\alpha(s)))ds \right| \le f(t)\left( |x(t)|^p + \int_0^t |K(s, x(\alpha(s)))| ds \right) \tag{3.3}$$

$$x_0 + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \left[ x_0^{p(1-p)} + (1-p)\int_0^s \left[ \frac{pf(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} + \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} \right] d\lambda \right]^{\frac{1}{p}} ds < \infty \tag{3.4}$$

where $f, g, x, p$ and $\alpha$ are defined as in Theorem2.3.Integrating both sides of (3.1) from 0 to $t$ ,we have

$$x(t) \le x_0 + \int_0^t \left[ M(s, x(\alpha(s)), \int_0^s H(\lambda, x(\alpha(\lambda)))d\lambda \right] ds, \ \forall t \in I \tag{3.5}$$

Using the conditions (3.2) and (3.3) in (3.5),we observe that

$$|x(t)| \le |x_0| + \int_0^t f(s)\left[ |x(s)|^p + \int_0^s g(\lambda)|x(\lambda)|^p d\lambda \right] ds,$$

$$|x(t)| \le |x_0| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \left[ |x(s)|^p + \int_0^s \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} |x(\lambda)|^p d\lambda \right] ds \tag{3.6}$$

Now a suitable application of the inequality given in Theorem2.3 with modifications to the above inequality leads to

$$|x(t)| \le x_0 + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \left[ x_0^{p(1-p)} + (1-p)\int_0^s \left[ \frac{pf(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} + \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} \right] d\lambda \right]^{\frac{1}{p}} ds$$

For all .thus from the hypotheses (3.4) and the estimation in (3.6) implies the boundedness of the solution of (3.1).

**Competing Interest:** The author declares that she has no competing interests.

## References

Abdeldaim, A, Yakout, M: On some new integral inequalities of Gronwall – Bellman -Pachpattetype, Appl.Math.Comput.217, 7887-7899, (2011).

Agarwal, RP, Deng,S, Zhang,W:Generalization of a retarded Gronwall-like inequality and its applications.Appl.Math.Comput.165, 599-612,(2005).

Bellman, R:,The stability of solutions of linear differential equations,Duke.Math.J.10.643- 647,(1943).

Bihari, IA: generalization of a lemma of Bellman and its applications to uniqueness problems of differential equations, Acta.Math.Acad.Sci.Hunger.7,71-94,(1956).

Cheung, WS:Some new nonlinear inequalities and applications to boundary value problems. Nonlinear.Anal.64, 2112-2128,(2006).

Gronwall, TH: Note on the derivatives with respect to a parameter of solutions of a system of differential equations, Ann.of Math.20.292-296,(1919).

Kim, YH:Onsome new integral inequalities for function in one and two variables,Acta.Math.Sinica,21423-434,(2005).

Lipovan, O: A retarded Gronwall-like inequality and its applications.J.Math.Anal.Appl.252, 389-401, (2000).

Pachpatte, BG: Inequalities for Differential and Integral Equations.Academic Press London,(1998).

Wang, WS, Shen,C:On a generalized retarded integral inequality with two variables.J.Inequal.Appl2008,9,(2008).

Wang, Ws, Shen, C:Some new generalized retarded nonlinear integral inequalities with iterated integrals and their application,J.Inequal.Appl 2012.