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Orbital Hausdorff Stability of the Solutions of Differential Equations with Variable Structure and Impulses

Katya G. Dishlieva[1]

Abstract

The concept of orbital gravitation for the systems of differential equations (SDE) with fixed structure and without impulses is introduced. The problems for non-autonomous nonlinear SDE with variable structure and impulses are the main object of investigation. Sufficient conditions for Hausdorff orbital stability of the solutions of such systems are found. The main requirement is any component of the systems to be orbital gravitating.

Keywords: variable impulsive effects, variable structure, stability, Hausdorff metric

1. Introduction

There are many continuous dynamic processes which are subjected to the "short-term" discrete external influences. As a result of these effects, they have a "break character" of development. Their dynamics are described (modeled) by the piecewise continuous functions. In general, these modeling functions are the solutions of impulsive differential equations (with fixed or variable impulsive moments). The approximation of such solutions by means of the smooth functions (such as algebraic polynomials, trigonometrical polynomials, etc.) on the uniform distance is not effective. The "significant differences" between discontinuous solutions and the smooth functions from the approximating class (in the terms of uniform distance) are the reason for this conclusion. These differences are "irreparably large" in sufficiently small surroundings of the breakpoints. In such cases it might use a considerably "weaker" metrics, such as the metrics based on a different integrated distances.

In some cases, taking into account the research objectives, the integral metrics is not adequate and therefore useless. On the other hand, the approximation of piecewise continuous solutions of the impulsive SDE using some classes of approximating piecewise smooth functions again in the terms of uniform distance is satisfactory only when the moments of impulses (breakpoints of the solutions) are fixed in advance. If we remove some "parts" of the solutions, we can approximate uniformly the solutions of impulsive equations with variable moments of impulsive effects. Usually, these parts are defined in the symmetrical surroundings of the impulsive moments. It is clear that such approximations are also not meaningful. Abovementioned problems could be overcome by using the Hausdorff distance between the trajectories of the studied solutions and the approximating functions. Research in this work are motivated by the notes made above. During the last years, a number of scientific papers are devoted to the qualitative theory of differential equations without impulses which use the Hausdorff metric see Ahmad and Sivasundaram (2006, 2008), Dishliev et al. (2011), Dishliev and Dishlieva (2011) and Dishlieva et al. (2014). Here, this metric is fundamental in the study of the solutions properties of non-autonomous systems of differential equations with variable structure and impulses. The qualitative research of SDE with variable impulsive effects is basic subject in a number of publications, we will mention: Akhmet (2005), Bainov and Dishliev (1989, 1997), Benchohra and Ouahab (2003) and Benchohra et al. (2004, 2005).

[1] PhD, Department of Differential Equations, Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Bulgaria; 8 "Kliment Ohridski", Sofia 1000, Bulgaria.
Phone: +359 895 581 455, E-mail: kgd@tu-sofia.bg, kgdishlieva@gmail.com

The interest towards these equations is determined by their many applications see Akhmet et al. (2006), Dishliev et al. (2011), Bainov et al. (1989, 1997), Gao et al. (2006, 2007), Nenov (1999), Nie and Peng et al. (2009) and Nie and Teng et al. (2009). In this paper, we study one specific class of non-linear non-autonomous SDE with variable structure and impulses. The change of the right hand sides of system and the impulsive effects are realized simultaneously at the moments in which the trajectory of corresponding initial value problem meets the consecutive "switching" set.

These sets are situated in the phase space of SDE. Our investigations are performed in the particular case for which the following assumptions are valid:

- the sets: $\{f_i;\ i=1,2,...\}$ of the right hand sides of SDE, $\{\Phi_i;\ i=1,2,...\}$ of the switching sets and $\{I_i;\ i=1,2,...\}$ of the impulsive functions consisting of a finite number of different elements:

$$\{f_i(t,x);\ i=1,2,...\}=\{\tilde{f}_j(t,x);\ j=1,2,...,k\},$$
$$\{\Phi_i;\ i=1,2,...\}=\{\tilde{\Phi}_j;\ j=1,2,...,k\},$$
$$\{I_i(x);\ i=1,2,...\}=\{\tilde{I}_j(x);\ j=1,2,...,k\};$$

- the switching sets coincide with the parts of predefined hyperplanes.

For the problem described above, the following terms are introduced:

- orbital gravitation;
- orbital Hausdorff stability on the initial condition.

These terms could be related for SDE without impulses and with a fixed structure. For instance, we will say that such a system is orbital gravitating with constant κ if the Hausdorff distance between its two arbitrary trajectories is κ times less than the Euclidean distance between them.m Let the right hand side of the system with variable structure and impulses consistently coincides with the functions $f_1, f_2, \ldots$. Assume that the solutions of each of the corresponding system without impulses $\frac{dx}{dt}=f_i(t,x),\ i=1,2,\ldots$, are orbital gravitating. Under this basic assumption, the sufficient conditions are found under which the solutions of the basic (studied) system with variable structure and impulses are orbital Hausdorff stable on the initial condition.

2. Preliminary Remarks

Further, we will use the following notations. Let the points $a(a^1,a^2,...,a^n),\ b(b^1,b^2,...,b^n)\in R^n$. Their scalar product is $\langle a,b\rangle=a^1b^1+a^2b^2+...+a^nb^n$. The Euclidean distance between these points is denoted by

$$\rho(a,b)=\|a-b\|=\sqrt{(a^1-b^1)^2+(a^2-b^2)^2+...+(a^n-b^n)^2}\ .$$

Let the nonempty sets $A,\ B\subset R^n$. Then the Euclidean and Hausdorff distance between them is denoted by

$$\rho_E(A,B)=\inf\{\inf\{\rho(a,b),\ a\in A\},\ b\in B\}$$

and

$$\rho_H(A,B)=\max\{\sup\{\inf\{\rho(a,b),\ a\in A\},\ b\in B\},\ \sup\{\inf\{\rho(a,b),\ b\in B\},\ a\in A\}\}.$$

If at least one of the sets A or B is empty, then for convenience we shall assume that

$\rho_E(A,B)=0$ and $\rho_H(A,B)=0$.

We will denote the contour and closure of the set A with ∂A and $\overline{A}$, respectively.

The following theorem will be used.

Theorem 1 *(Dishlieva et. al. (2014)). Assume that the sets $A_1, A_2, ..., A_k, B_1, B_2, ..., B_k \subset R^n$ are bounded. Then*

$$\rho_H\left(A_1 \cup A_2 \cup ...A_k,\ B_1 \cup B_2 \cup ...B_k\right) \le \max\left\{\rho_H\left(A_1, B_1\right), \rho_H\left(A_2, B_2\right), ..., \rho_H\left(A_k, B_k\right)\right\}.$$

We will define the concept of orbital gravitating SDE. Consider the initial value problem

$$\frac{dx}{dt} = f\left(t, x\right), \quad x\left(t_0\right) = x_0,$$

where: function $f : R^+ \times D \to R^n$; set D is a domain of R^n; the point $\left(t_0, x_0\right) \in R^n \times D$. The solution of this initial value problem will be denoted by $X\left(t; t_0, x_0\right)$ and the corresponding trajectory of the solution will be $\Gamma\left[t_0, \infty\right)$, i.e. $\Gamma\left[t_0, \infty\right) = \left\{X\left(t; t_0, x_0\right),\ t_0 \le t < \infty\right\}$.

The solution and trajectory of the perturbed initial value problem $\frac{dx}{dt} = f\left(t, x\right),\ x\left(t_0^*\right) = x_0^*$, where the point $\left(t_0^*, x_0^*\right) \in R^n \times D$, we will be denoted by $X^*\left(t; t_0^*, x_0^*\right)$ and $\Gamma^*\left[t_0^*, \infty\right)$, respectively.

We introduce the definitional equations for the Euclidean, Hausdorff and uniform distance, between the trajectories of the above mentioned problems, respectively:

$$\rho_E\left(\Gamma^*\left[t_0^*, \infty\right), \Gamma\left[t_0, \infty\right)\right) = \inf\left\{\inf\left\{\rho\left(X^*\left(t; t_0^*, x_0^*\right),\ X\left(t; t_0, x_0\right)\right),\ t_0^* \le t^* < \infty\right\},\ t_0 \le t < \infty\right\};$$

$$\rho_H\left(\Gamma^*\left[t_0^*, \infty\right), \Gamma\left[t_0, \infty\right)\right) = \max\left\{\sup\left\{\inf\left\{\rho\left(X^*\left(t; t_0^*, x_0^*\right),\ X\left(t; t_0, x_0\right)\right),\ t_0^* \le t^* < \infty\right\},\ t_0 \le t < \infty\right\},\right.$$
$$\left.\sup\left\{\inf\left\{\rho\left(X^*\left(t; t_0^*, x_0^*\right),\ X\left(t; t_0, x_0\right)\right),\ t_0 \le t < \infty\right\},\ t_0^* \le t^* < \infty\right\}\right\};$$

$$\rho_R\left(\Gamma^*\left[t_0, \infty\right), \Gamma\left[t_0, \infty\right)\right) = \sup\left\{\rho\left(X^*\left(t; t_0, x_0^*\right),\ X\left(t; t_0, x_0\right)\right),\ t_0 \le t < \infty\right\}.$$

It is natural to assume that $t_0^* = t_0$ for the uniform distance.

Definition 1. *We will say that the system considered is orbital gravitating in the domain D with a constant $\kappa \ge 1$, if:*

$$\left(\forall t_0^*, t_0 \in D\right)\left(\forall x_0^*, x_0 \in D\right) \Rightarrow \rho_H\left(\Gamma^*\left[t_0^*, \infty\right), \Gamma\left[t_0, \infty\right)\right) \le \kappa.\rho_E\left(\Gamma^*\left[t_0^*, \infty\right), \Gamma\left[t_0, \infty\right)\right)$$
$$\Leftrightarrow \max\left\{\sup\left\{\inf\left\{\rho_E\left(X^*\left(t^*; t_0^*, x_0^*\right), X\left(t; t_0, x_0\right)\right),\ t \ge t_0\right\},\ t^* \ge t_0^*\right\},\right.$$
$$\left.\sup\left\{\inf\left\{\rho_E\left(X^*\left(t^*; t_0^*, x_0^*\right), X\left(t; t_0, x_0\right)\right),\ t^* \ge t_0^*\right\},\ t \ge t_0\right\}\right\}$$
$$\le \kappa.\inf\left\{\inf\left\{\rho_E\left(X^*\left(t^*; t_0^* x_0^*\right),\ X\left(t; t_0, x_0\right)\right),\ t \ge t_0\right\},\ t^* \ge t_0^*\right\}.$$

The main object of investigation is the following initial value problem

$$\frac{dx}{dt} = f_i\left(t, x\right),\ \left\langle a_i, x\left(t\right)\right\rangle \ne \alpha_i; \tag{1}$$

$$x\left(t+0\right) = x\left(t\right) + I_i\left(x\left(t\right)\right),\ \left\langle a_i, x\left(t\right)\right\rangle = \alpha_i,\ i = 1, 2, ...; \tag{2}$$

$x(t_0) = x_0$, (3)

where: the dimensionality is $n \in N$; set D is a nonempty domain in R^n; the functions $f_i : R^+ \times D \to R^n$; the vectors $a_i = (a_i^1, a_i^2, ..., a_i^n) \in R^n$, $\|a_i\| = 1$; the constants $\alpha_i \in R$; the functions $I_i : \Phi_i \to R^n$, where $\Phi_i = \{x \in D; \langle a_i, x\rangle = \alpha_i\}$; the initial point $(t_0, x_0) \in R^+ \times D$.

The sets Φ_i, $i = 1, 2, \dots$, are called switching sets. They are parts of the hyperplanes, situated in D. The functions I_i, $i = 1, 2, \dots$, are named impulsive functions. Assume that $(Id + I_i) : \Phi_i \to D$ is fulfilled, where Id is an identity in R^n. The moments in which the trajectory of initial value problem above meets consistently the switching sets $\Phi_1, \Phi_2, \dots$ are denoted by $t_1, t_2, \dots$. The inequalities $t_0 < t_1 < t_2 < \dots$ are fulfilled.

The solution $x(t; t_0, x_0)$ of the problem studied is a piecewise continuous function. We have:

1.1. For $t_0 \le t < t_1$, the solution of problem (1), (2), (3) coincides with the solution of problem (without impulses) (1), (3) for $i = 1$. The next inequality is valid

$\langle a_1, x(t; t_0, x_0)\rangle \ne \alpha_1$;

1.2. At the moment t_1, the following equalities are fulfilled:

$x(t_1; t_0, x_0) = x(t_1 - 0; t_0, x_0) = x_1$ and $\langle a_1, x(t_1; t_0, x_0)\rangle = \langle a_1, x_1\rangle = \alpha_1$;

1.3. At the moment $t_1 + 0$, we have

$x(t_1 + 0; t_0, x_0) = x(t_1; t_0, x_0) + I_1(x(t_1; t_0, x_0)) = (Id + I_1)(x(t_1; t_0, x_0)) = (Id + I_1)(x_1) = x_1^+$;

2.1. For $t_1 < t < t_2$, the solution of problem (1), (2), (3) coincides with the solution of system (1) for $i = 2$ with the initial condition $x(t_1 + 0) = x_1^+$. The next inequality is valid

$\langle a_2, x(t; t_0, x_0)\rangle \ne \alpha_2$;

2.2. At the moment t_2, the following equalities are satisfied:

$x(t_2; t_0, x_0) = x(t_2 - 0; t_0, x_0) = x_2$ and $\langle a_2, x(t_2; t_0, x_0)\rangle = \langle a_2, x_2\rangle = \alpha_2$;

2.3. Furthermore,

$x(t_2 + 0; t_0, x_0) = x(t_2; t_0, x_0) + I_2(x(t_2; t_0, x_0)) = (Id + I_2)(x(t_2; t_0, x_0)) = (Id + I_2)(x_2) = x_2^+$, etc.

Along the problem (1), (2), (3), we consider the corresponding perturbed problem (1), (2) with the initial condition

$x(t_0^*) = x_0^*$, (4)

where the initial point $(t_0^*, x_0^*) \in R^+ \times D$. The solution of perturbed problem (1), (2), (4) will be denoted by $x^*(t; t_0^*, x_0^*)$ and the moments at which the trajectory of this initial value problem meets the switching sets $\Phi_1, \Phi_2, \dots$ by $t_1^*, t_2^*, \dots$ respectively. The inequalities $t_0^* < t_1^* < t_2^* < \dots$ are valid.

Also, we will use the following notations:

$$x_i = x(t_i;t_0,x_0) = x(t_i - 0;t_0,x_0),$$
$$x_i^* = x^*(t_i^*;t_0^*,x_0^*) = x^*(t_i^* - 0;t_0^*,x_0^*),$$
$$x_i^+ = x(t_i;t_0,x_0) + I_i(x(t_i;t_0,x_0)) = (Id + I_i)(x_i) = x(t_i + 0;t_0,x_0),$$
$$x_i^{*+} = x^*(t_i^*;t_0^*,x_0^*) + I_i(x^*(t_i^*;t_0^*,x_0^*)) = (Id + I_i)(x_i^*) = x^*(t_i^* + 0;t_0^*,x_0^*),\ i = 1,2,\ldots.$$

The trajectories of problems (1), (2), (3) and (1), (2), (4) are denoted respectively by:
$\gamma[t_0,\infty) = \{x(t;t_0,x_0),\ t_0 \le t < \infty\}$ and $\gamma^*[t_0^*,\infty) = \{x^*(t;t_0^*,x_0^*),\ t_0^* \le t < \infty\}$.

For each $i = 1,2,\ldots$, the solutions and trajectories of the problems without impulses (1), (3) and (1), (4) are denoted respectively by $X_i(t;t_0,x_0),\ \Gamma_i[t_0,\infty)$ and $X_i^*(t;t_0^*,x_0^*),\ \Gamma_i^*[t_0^*,\infty)$.

The notations for the parts of trajectories $\gamma[t_0,\infty),\ \gamma^*[t_0^*,\infty),\ \Gamma_i[t_0,\infty),\ \Gamma_i^*[t_0^*,\infty)$, which are defined for $0 \le \theta' \le t \le \theta''$ are:

$$\gamma[\theta',\theta''],\ \gamma^*[\theta',\theta''],\ \Gamma_i[\theta',\theta''],\ \Gamma_i^*[\theta',\theta''].$$

Definition 2. *We will say that the solution of problem (1), (2), (3) is an orbital Hausdorff stable on the initial condition (initial point) if:*

$$(\forall \varepsilon > 0)\ (\forall (t_0,x_0) \in R^+ \times D)(\exists \delta = \delta(\varepsilon,t_0,x_0) > 0):$$
$$(\forall t_0^* \in R^+, |t_0^* - t_0| < \delta)(\forall x_0^* \in D, \|x_0^* - x_0\| < \delta) \Rightarrow\ \rho_H(\gamma^*[t_0^*,\infty),\gamma[t_0,\infty)) < \varepsilon.$$

The main objective in the paper is to find the sufficient conditions for the Hausdorff orbital stability on the initial condition of the solution of problem (1), (2), (3). We assume that the successive changes in the structure (right hand side of the system); the impulsive functions and the switching sets are restricted to a finite number of options. Consider that there are k eligible options. More precisely, we suppose that the following condition is valid:

H0. *There exist: right hand sides of the system* $\tilde{f}_j : R^+ \times D \to R^n$*; the vectors* $\tilde{a}_j \in R^n$, $\|\tilde{a}_j\| = 1$*; the constants* $\tilde{\alpha}_j \in R$*; the switching sets* $\tilde{\Phi}_j = \{x \in D;\ \langle \tilde{a}_j, x\rangle = \tilde{\alpha}_j\}$ *and the impulsive functions* $\tilde{I}_j : \tilde{\Phi}_j \to R^n$, $j = 1,2,\ldots,k$, *such that*

$$(\forall i = 1,2,\ldots)(\exists j_i \in \{1,2,\ldots,k\}):$$
$$f_i(t,x) = \tilde{f}_{j_i}(t,x),\ (t,x) \in R^+ \times D;\ a_i = \tilde{a}_{j_i};\ \alpha_i = \tilde{\alpha}_{j_i};\ \Phi_i = \tilde{\Phi}_{j_i};\ I_i(x) = \tilde{I}_{j_i}(x),\ x \in \tilde{\Phi}_{j_i}.$$

Further, we assume that for each $j = 1,2,\ldots,k$, the following conditions are fulfilled:

H1. *The functions* $\tilde{f}_j \in C[R^+ \times D,\ R^n]$ *and* $\tilde{f}_j \in Lip_x[R^+ \times D,\ R^n]$.

H2. *There exist positive constants* C_j^1 *such that*

$$(\forall (t,x) \in R^+ \times D) \Rightarrow \|\tilde{f}_j(t,x)\| \le C_j^1.$$

H3. *For every point* $(t_0,x_0) \in R^+ \times D$, *the initial value problem without impulses* $\frac{dx}{dt} = \tilde{f}_j(t,x),\ \ x(t_0) = x_0$ *possesses the unique solutions which are defined for* $t \ge t_0$.

H4. *The switching sets* $\tilde{\Phi}_j$ *are bonded and following inclusions are valid* $\overline{(\tilde{\Phi}_j)} \setminus \tilde{\Phi}_j \subset \bar{D} \setminus D$.

H5. *There exist constants* $C_j^2,\ 0<C_j^2\le 1$, *and function* $\varphi,\ \varphi\in C\left[\overline{D},R\right],\ \varphi(x)>0$ *for* $x\in\overline{\left(\tilde{\Phi}_j\right)}$, *such that*
$\left(\forall(t,x)\in R^+\times\tilde{\Phi}_j\right)\ \Rightarrow\ \left|\left\langle\tilde{a}_j,\tilde{f}_j(t,x)\right\rangle\right|\ge C_j^2\left\|\tilde{f}_j(t,x)\right\|\ge\varphi(x)$.

H6. *There exist positive constants* C_j^3 *such that the next inequalities are valid*
$\rho_E\left(\tilde{\Phi}_i,\left(Id+\tilde{I}_j\right)\left(\tilde{\Phi}_j\right)\right)\ge C_j^3,\quad i=1,2,...,k$.

H7. *There exist positive constants* C_j^4 *such that*
$\left(\forall x',x''\in\tilde{\Phi}_j\right)\ \Rightarrow\ \rho_E\left(x'+\tilde{I}_j(x'),x''+\tilde{I}_j(x'')\right)\le C_j^4.\rho_E(x',x'')$.

Remark 1. Using condition H0, we deduce that the conditions H1 - H7 are satisfied if we replace $\tilde{f}_j,\ \tilde{\Phi}_j,\ \tilde{a}_j,\ \tilde{I}_j,\quad j=1,2,...,k$, respectively by $f_i,\Phi_i,\ a_i,I_i,\ i=1,2,...$.

In the next theorem, the sufficient conditions for the absence of condensation at the switching moments are found, i.e. the conditions under which $\lim\limits_{i\to\infty}t_i=\infty$.

***Theorem 2**. Assume that:*
1. *The conditions H0, H1, H2, H3 and H6 hold.*
2. *The trajectory of problem (1), (2), (3) meets infinitely many switching sets* $\Phi_1,\Phi_2,...$ *at the switching moments* $t_1,t_2,...$, *respectively.*

Then $\lim\limits_{i\to\infty}t_i=\infty$.

Proof. We evaluate below the difference (t_2-t_1), i.e. the difference between the second and first switching moment of basic problem (1), (2), (3). According to the conditions H6 and H0 and since $x_2=x(t_2;t_0,x_0)\in\Phi_2$ and $x_1^+\in(Id+I_1)(\Phi_1)$, we find that

$$C_{j_1}^3\le\rho_E\left(\tilde{\Phi}_{j_2},\left(Id+\tilde{I}_{j_1}\right)\left(\tilde{\Phi}_{j_1}\right)\right)=\rho_E\left(\Phi_2,(Id+I_1)(\Phi_1)\right)\le\rho_E\left(\Phi_2,x_1^+\right)\le\rho_E\left(x_2,x_1^+\right)$$
$$=\left\|x_2-x_1^+\right\|=\left\|x(t_2;t_0,x_0)-x(t_1+0;t_0,x_0)\right\|=\left\|\int_{t_1}^{t_2}f_2\left(x\left(\tau;t_1,x_1^+\right)\right)d\tau\right\|\le\int_{t_1}^{t_2}\left\|\tilde{f}_{j_2}\left(x\left(\tau;t_1,x_1^+\right)\right)\right\|d\tau$$
$$\le C_{j_2}^1(t_2-t_1).$$

Therefore, $t_2-t_1\ge C_{j_1}^3\big/C_{j_2}^1$. Similarly, using that the switching moments are innumerable, we obtain the estimates
$t_{i+1}-t_i\ge C_{j_i}^3\big/C_{j_{i+1}}^1,\quad i=1,2,...$.

From the above, it follows that

$$\lim_{i\to\infty}t_i=t_1+\lim_{i\to\infty}(t_i-t_1)=t_1+\lim_{i\to\infty}\left((t_i-t_{i-1})+(t_{i-1}-t_{i-1})+...+(t_2-t_1)\right)$$
$$\ge t_1+\lim_{i\to\infty}\left(\frac{C_{j_{i-1}}^3}{C_{j_i}^1}+\frac{C_{j_{i-2}}^3}{C_{j_{i-1}}^1}+...+\frac{C_{j_1}^3}{C_{j_2}^1}\right)\ge t_1+\frac{\min\left\{C_j^3;\quad j=1,2,...,k\right\}}{\max\left\{C_j^1;\quad j=1,2,...,k\right\}}\lim_{i\to\infty}(i-1)=\infty.$$

The Theorem is proved.

Using Theorem 2, we deduce that the solution of the problem (1), (2), (3) is continuable up to infinity independently of the choice of initial point x_0 of the set $D \setminus \Phi_1$.

Theorem 3. *Assume that the conditions H0, H1, H2, H3 and H6 are satisfied.*

Then for each point $(t_0, x_0) \in R^+ \times (D \setminus \Phi_1)$*, the solution of problem (1), (2), (3) exists and is unique for* $t \ge t_0$.

In particular, from the proposition above, we obtain that $(\forall (t_0, x_0) \in R^+ \times D) \Rightarrow \gamma[t_0, \infty) \in D$.

The case when the trajectory of problem (1), (2), (3) does not intersect the first switching set is studied in the next theorem. In this case, the basic system degenerates into a system with a fixed structure and without impulses.

Theorem 4. *Assume that:*

1. The conditions H0, H1, H3, H4 are fulfilled for $j = j_1$.

2. The system $\dfrac{dx}{dt} = f_1(t, x) \Leftrightarrow \dfrac{dx}{dt} = f_{j_1}(t, x)$ *is orbital gravitating in domain* D *with constant* $\kappa_{j_1} \ge 1$.

3. The trajectory $\gamma[t_0, \infty)$ *does not intersect the switching set* $\Phi_1 = \Phi_{j_1}$.

Then for every initial point $(t_0, x_0) \in R^+ \times D, \;\; x_0 \notin \Phi_1$*, the solution of problem (1), (2), (3) (in the case, the solution of basic initial value problem coincides with the solution of problem (1), (3) for* $i = 1$*) is an orbital Hausdorff stable on the initial condition.*

Proof. Let ε be an arbitrary positive constant.

Since the trajectory $\gamma[t_0, \infty)$ does not cross the switching set Φ_1, then $\gamma[t_0, \infty) = \Gamma_1[t_0, \infty)$. Assume that $\rho_E(\gamma[t_0, \infty), \overline{\Phi_1}) = 0$. According to condition H4 we have that the set $\overline{\Phi_1}$ is compact. As $\gamma[t_0, \infty)$ is a closed set, we conclude that $\gamma[t_0, \infty) \cap \overline{\Phi_1} \ne \varnothing$, whence we find that $\gamma[t_0, \infty) \cap \;\; \cap \overline{\Phi_1} \setminus \Phi_1 \ne \varnothing$. Taking into account the inclusion $\overline{\Phi_1} \setminus \Phi_1 \subset \overline{D} \setminus D$, we deduce that $\gamma[t_0, \infty) \cap \;\; \cap \overline{D} \setminus D \ne \varnothing$. In other words, the trajectory of problem (1), (3) meets the contour of domain D, i.e. the solution of problem without impulses is not continuable after a certain moment. Therefore, the condition H3 is not satisfied. Thus, we derive the validity of inequality $\rho_E(\gamma[t_0, \infty), \overline{\Phi_1}) > 0$. Suppose that

$$\rho_E(\gamma[t_0, \infty), \Phi_1) = r, \tag{5}$$

where r is a positive constant. Let $\delta = \dfrac{1}{\kappa_{j_1}} \min(\varepsilon, r)$. Given that system (1) is orbital gravitating for $i = 1$ in domain D with the coefficient $\kappa_{j_1} \ge 1$, we conclude that for any initial point $x_0^* \in D$, for which $\left\| x_0^* - x_0 \right\| < \delta$, it is fulfilled

$$\rho_H\left(\Gamma_1^*[t_0^*, \infty),\, \Gamma_1[t_0, \infty)\right) \le \kappa_{j_1} . \rho_E\left(\Gamma_1^*[t_0^*, \infty),\, \Gamma_1[t_0, \infty)\right) \tag{6}$$

$$\le \kappa_{j_1} . \rho_E(x_0^*, x_0) = \kappa_{j_1} . \left\| x_0^* - x_0 \right\| < \kappa_{j_1} . \delta = \min(\varepsilon, r).$$

Taking into consideration equality (5) and evaluation (6), we find

$$\rho_H\left(\Gamma_1^*[t_0^*, \infty), \Phi_1\right) \ge -\rho_H\left(\Gamma_1^*[t_0^*, \infty),\; \Gamma_1[t_0, \infty)\right) + \rho_H\left(\Gamma_1[t_0, \infty), \Phi_1\right) > -r + r = 0.$$

Therefore, the trajectory $\Gamma_1^*\left[t_0^*,\infty\right)$ of perturbed problem without impulses (1), (4) (for $i=1$) does not meet switching set Φ_1, which means that $\gamma^*\left[t_0^*,\infty\right)=\Gamma_1^*\left[t_0^*,\infty\right)$. In other words, the perturbed solution is not subjected to the impulsive effects (as, incidentally, the solution of basic problem). Using this conclusion and evaluation (6), we obtain that

$$\left(\forall x_0^*\in D,\ \rho_E\left\|x_0^*-x_0\right\|<\delta\right)\Rightarrow\ \rho_H\left(\gamma^*\left[t_0^*,\infty\right),\gamma\left[t_0,\infty\right)\right)=\rho_H\left(\Gamma_1^*\left[t_0^*,\infty\right),\Gamma_1\left[t_0,\infty\right)\right)<\varepsilon,$$

i.e. the solution of problem (1), (2), (3) is an orbital Hausdorff stable on the initial point.

The Theorem is proved.

3. Basic Result

Theorem 5. *Assume that:*

1. The conditions H0 - H7 are satisfied.

2. The systems $\dfrac{dx}{dt}=\tilde{f}_j(t,x)$ *are orbital gravitating in domain* D *with the constants* $\kappa_j\geq 1,\ j=1,2,\dots,k$*, respectively.*

3. The trajectory $\gamma\left[t_0,\infty\right)$ *intersects the switching set* $\Phi_1=\tilde{\Phi}_{j_1}$.

4. The following inequalities are valid $C_j^4<\dfrac{C_j^2}{\kappa_j\left(1+C_j^2\right)},\ \ j=1,2,\dots,k$.

Then for every initial point $\left(t_0,x_0\right)\in R^+\times D,\ \ x_0\notin\Phi_1,$ *the solution of basic problem (1), (2), (3) is orbital Hausdorff stable on the initial value condition.*

Proof. Using Theorem 4, we assume that the trajectory $\gamma\left[t_0,\infty\right)$ intersects consecutively (repeatedly, including infinitely many times) the switching sets $\Phi_1,\Phi_2,\dots$.

The proof of this theorem is separated into several parts:

Part 1. We will show that if the difference $\left|t_0^*-t_0\right|$ between the initial moments t_0^* and t_0 on the one hand and the Euclidean distance $\rho_E\left(x_0^*,x_0\right)=\left\|x_0^*-x_0\right\|$ between the initial points x_0^* and x_0 on the other hand are "sufficiently small", then the trajectory $\gamma^*\left[t_0^*,\infty\right)$ of perturbed problem (1), (2), (4) also intersects the switching set $\Phi_1=\tilde{\Phi}_{j_1}$. According to condition H3, the solutions $X_1\left(t;t_0,x_0\right)$ and $X_1^*\left(t;t_0,x_0^*\right)$ of the problems without impulses (1), (3) and (1), (4) (for $i=1$) are defined for every $t\geq t_0$ and $t\geq t_0^*$, respectively. These solutions are not subjected to the impulsive effects (regardless of the presence or absence of the meetings with the switching set Φ_1).

By condition H5, it follows that for each point $(t,x)\in R^+\times\tilde{\Phi}_{j_1}$, the scalar product $\left\langle\tilde{a}_{j_1},\tilde{f}_{j_1}(t,x)\right\rangle$ has a permanent sign. Further, we suppose that $\left\langle\tilde{a}_{j_1},\tilde{f}_{j_1}(t,x)\right\rangle>0,\ (t,x)\in R^+\times\tilde{\Phi}_{j_1}$.

There are similar considerations in the other case. Using condition H5 and the additional clarification made above, we obtain that

$$\left\langle a_1,f_1(t,x)\right\rangle=\left\langle\tilde{a}_{j_1},\tilde{f}_{j_1}(t,x)\right\rangle\geq C_{j_1}^2\left\|\tilde{f}_{j_1}(t,x)\right\|\geq C_{j_1}^2\varphi(x)>0,\ \ (t,x)\in R^+\times\tilde{\Phi}_{j_1}.$$

Assume that $\mu,\ 0<\mu<1$, is an arbitrary constant which value will be specified further. From the inequality above, we deduce that

$$\left\langle \vec{a}_{j_1}, \vec{f}_{j_1}(t,x)\right\rangle - C_{j_1}^2 \left\|\vec{f}_{j_1}(t,x)\right\| \geq (1-\mu)C_{j_1}^2 \left\|\vec{f}_{j_1}(t,x)\right\| \geq (1-\mu)C_{j_1}^2\varphi(x)>0,\quad (t,x)\in R^+\times\tilde{\Phi}_{j_1}.$$

Using again condition H5, we derive that

$$\left(\exists C_{j_1}^5 = const>0\right):\left(\forall x\in\Phi_{j_1}\right)\Rightarrow \varphi(x)\geq C_{j_1}^5.$$

Then

$$\left\langle \vec{a}_{j_1}, \vec{f}_{j_1}(t,x)\right\rangle - C_{j_1}^2 \left\|\vec{f}_{j_1}(t,x)\right\| \geq (1-\mu)C_{j_1}^2 C_{j_1}^5 >0,\quad (t,x)\in R^+\times\tilde{\Phi}_{j_1}.$$

Since:

- left hand side of the latter inequality is continuous function;
- this function is greater than a positive constant over the switching set $\tilde{\Phi}_{j_1}$;
- switching set $\tilde{\Phi}_{j_1}$ is bounded, it follows that there exists a positive constant $\Delta=\Delta(\mu)$ such that for every constant $d,\ 0\leq d\leq\Delta$, the following relations are satisfied:
- the initial point $x_0\notin B(\Phi_1,d)=\left\{x\in D,\ \alpha_1-d\leq\langle a_1,x\rangle\leq\alpha_1+d\right\}$;
- the next inequalities are valid

$$\left\langle a_1, f_1(t,x)\right\rangle \geq \mu C_{j_1}^2\left\|f_1(t,x)\right\| \text{ and } \left\|f_1(t,x)\right\|>C_{j_1}^6,\quad (t,x)\in R^+\times B(\Phi_1,d), \tag{7}$$

where $C_{j_1}^6$ is a sufficiently small positive constant. Set $B(\Phi_1,d)$ consists of all the points belonging to phase space D, which are situated between the hyperplanes:

$$\Phi_{1-d}=\left\{x\in R^n;\langle a_1,x\rangle=\alpha_1-d\right\} \text{ and } \Phi_{1+d}=\left\{x\in R^n;\langle a_1,x\rangle=\alpha_1+d\right\}.$$

Let $t_1>0$ be the moment of the first meeting between the trajectory $\gamma[t_0,\infty)$ and switching set Φ_1. Consider a function $A(t)=\left\langle a_1, X_1(t;t_0,x_0)\right\rangle-\alpha_1,\quad t\geq t_0$. It is fulfilled

$$A(t_1)=\left\langle a_1, X_1(t_1;t_0,x_0)\right\rangle-\alpha_1=\left\langle a_1, x(t_1;t_0,x_0)\right\rangle-\alpha_1=\left\langle a_1,x_1\right\rangle-\alpha_1=0. \tag{8}$$

This means that point $X_1(t_1;t_0,x_0)\in\Phi_1\in B(\Phi_1,d)$. On the other hand, since the point $x_0=X_1(t_0;t_0,x_0)\notin B(\Phi_1,d)$, it follows that there exists a point $t_{-d},\ t_0<t_{-d}<t_1$, such that $X_1(t_{-d};t_0,x_0)\in\Phi_{1-d}\cap D$.

Using first estimate of (7), we obtain that for every point $t\geq t_1$, with property $-d\leq A(t)\leq d$, i.e. for which $X_1(t;t_0,x_0)\in B(\Phi_1,d)$, the next inequality takes place

$$\frac{d}{dt}A(t)=\frac{d}{dt}\left(\left\langle a_1, X_1(t;t_0,x_0)\right\rangle-\alpha_1\right) \tag{9}$$

$$=\left\langle a_1, f_1\left(X_1(t;t_0,x_0)\right)\right\rangle \geq \mu C_{j_1}^2\left\|f_1\left(X_1(t;t_0,x_0)\right)\right\| \geq \mu C_{j_1}^2 C_{j_1}^5=const>0.$$

From (8) and (9), it follows that there exists a point $t_{+d}>t_1$, such that $A(t_{+d})=d$, i.e. $X_1(t_{+d};t_0,x_0)\in\Phi_{1+d}\cap D$ (fig. 1).

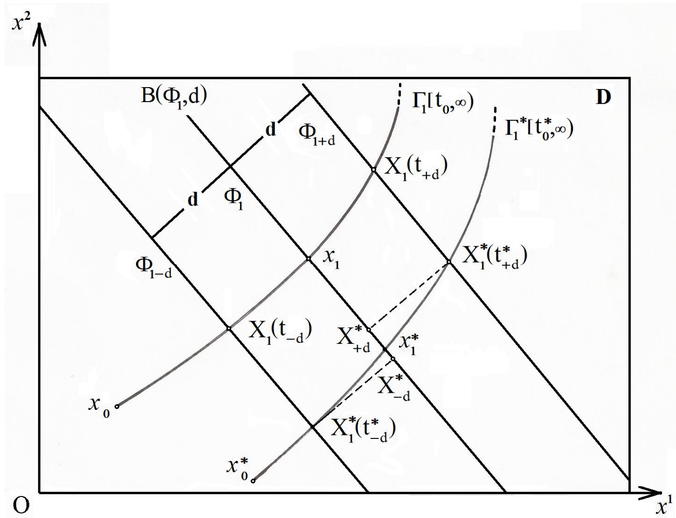

Fig. 1

The following notations are used in the figure:

$X_1(t_{-d}) = X_1(t_{-d};t_0,x_0),\ X_1(t_{+d}) = X_1(t_{+d};t_0,x_0),$

$X_1^*(t_{-d}^*) = X_1^*(t_{-d}^*;t_0^*,x_0^*),\ \ X_1^*(t_{+d}^*) = X_1^*(t_{+d}^*;t_0^*,x_0^*).$

We conclude that:

$$(\forall \mu,\ 0<\mu<1)(\exists \Delta = \Delta(\mu)>0):(\forall d,\ 0\le d\le \Delta)(\forall x_0 \in D\setminus B(\Phi_1,d))$$

$$\Rightarrow(\ \exists\ t_{-d},\ t_{+d}\in R,\ \ 0<t_{-d}<t_1<t_{+d}):\ X_1(t_{-d};t_0,x_0)\in \Phi_{1-d} \ \textit{ and }$$

$$X_1(t_{+d};t_0,x_0)\in \Phi_{1+d}.$$

Since $\|a_1\|=1$, then it is fulfilled $d=\rho_E(\Phi_1,\Phi_{1-d})=\rho_E(\Phi_1,\Phi_{1+d})>0$. According to condition 2 of this theorem, system (1) (for $i=1$) is an orbital gravitating in domain D with the coefficient $\kappa_{j_1}\ge 1$. Let $d<\min(\varepsilon,r)$ and $\delta = d/\kappa_{j_1}$. Using the inequality $\|x_0^*-x_0\| = \rho(x_0^*,x_0) < d/\kappa_{j_1}$ and that point $X_1(t_{-d};t_0,x_0)\in\Phi_{1-d}$, it follows that

$$\rho_E(\Gamma_1^*[t_0^*,\infty),\Phi_{1-d}) \le \rho_E(\Gamma_1^*[t_0^*,\infty),X_1(t_{-d};t_0,x_0)) \qquad (10)$$

$$\le \sup\{\rho_E(\Gamma_1^*[t_0^*,\infty),X_1(t;t_0,x_0)),\ t\ge 0\} = \sup\{\inf\{\rho_E(X_1^*(t^*;t_0^*,x_0^*),X_1(t;t_0,x_0)),\ t^*\ge 0\},\ t\ge 0\}$$

$$\le \rho_H(\Gamma_1^*[t_0^*,\infty),\Gamma_1[t_0,\infty)) \le \kappa_{j_1}\rho_E(\Gamma_1^*[t_0^*,\infty),\Gamma_1[t_0,\infty)) \le \kappa_{j_1}\rho_E(x_0^*,x_0) \le \kappa_{j_1}.\frac{d}{\kappa_{j_1}} = d.$$

Similarly, it is obtained

$$\rho_E(\Gamma_1^*[t_0^*,\infty),\Phi_{1+d}) \le d. \qquad (11)$$

From the inequalities (10) and (11), it follows that there exist the points t_{-d}^* and t_{+d}^*, $t_0^*<t_{-d}^*<t_{+d}^*$, such that the next estimates are valid:

$\rho_E\left(X_1^*\left(t_-^*;t_0^*,x_0^*\right),\Phi_{1-d}\right)<d$ and $\rho_E\left(X_1^*\left(t_+^*;t_0^*,x_0^*\right),\Phi_{1+d}\right)<d$. (12)

Consider a function $A^*(t)=\left\langle a_1, X_1^*\left(t;t_0^*,x_0^*\right)\right\rangle-\alpha_1,\quad t\geq 0$. Assume that X_{-d}^* and X_{+d}^* are orthogonal projections of the points $X_1^*\left(t_{-d}^*;t_0^*,x_0^*\right)$ and $X_1^*\left(t_{+d}^*;t_0^*,x_0^*\right)$, respectively on the hyperplane Φ_1. By (12), it follows that:

$\left(X_{-d}^*-X_1^*\left(t_{-d}^*;t_0^*,x_0^*\right)\right)$ and $\left(X_1^*\left(t_{+d}^*;t_0^*,x_0^*\right)-X_{+d}^*\right)$ are one-way vectors. Consider the case when vector a_1 has the same direction. The other case is considered similarly. Moreover, under the assumptions made at the beginning of Part 1 of this proof, i.e. if the inequality below is valid

$\left\langle a_{j_1}, f_{j_1}(t,x)\right\rangle>0,\quad (t,x)\in R^+\times\Phi_{j_1}\;\Leftrightarrow\;\left\langle a_1, f_1(t,x)\right\rangle>0,\quad (t,x)\in R^+\times\Phi_1$,

it can be shown that the vectors:

$a_1,\;\left(X_{-d}^*-X_1^*\left(t_{-d}^*;t_0^*,x_0^*\right)\right),\;\left(X_1^*\left(t_{+d}^*;t_0^*,x_0^*\right)-X_{+d}^*\right)$ have one and the same direction. Then the next inequalities are valid:

$$A^*\left(t_{-d}^*\right)=\left\langle a_1, X_1^*\left(t_{-d}^*;t_0^*,x_0^*\right)\right\rangle-\alpha_1=\left\langle a_1, X_{-d}^*\right\rangle+\left\langle a_1, X_1^*\left(t_{-d}^*;t_0^*x_0^*\right)-X_{-d}^*\right\rangle-\alpha_1$$
$$=-\left\langle a_1, X_{-d}^*-X_1^*\left(t_{-d}^*;t_0^*\,x_0^*\right)\right\rangle=-\left\|X_{-d}^*-X_1^*\left(t_{-d}^*;t_0^*\,x_0^*\right)\right\|<0;$$
$$A^*\left(t_{+d}^*\right)=\left\langle a_1, X_1^*\left(t_{+d}^*;t_0^*,x_0^*\right)\right\rangle-\alpha_1=\left\langle a_1, X_{+d}^*\right\rangle+\left\langle a_1, X_1^*\left(t_{+d}^*;t_0^*x_0^*\right)-X_{+d}^*\right\rangle-\alpha_1$$
$$=\left\langle a_1, X_1^*\left(t_{+d}^*;t_0^*\,x_0^*\right)-X_{+d}^*\right\rangle=\left\|X_1^*\left(t_{+d}^*;t_0^*\,x_0^*\right)-X_{+d}^*\right\|>0.$$

From the continuity of function A^* and both inequalities above, it follows that there exists a point $t_1^*,\quad 0<t_{-d}^*<t_1^*<t_{+d}^*$, such that

$A^*\left(t_1^*\right)=0\Leftrightarrow\left\langle a_1, X_1^*\left(t_1^*;t_0^*,x_0^*\right)\right\rangle-\alpha_1=0\;\Leftrightarrow\;X_1^*\left(t_1^*;t_0^*x_0^*\right)\in\Phi_1$,

i.e. trajectory $\Gamma_1^*\left[t_0^*,\infty\right)$ (respectively trajectory $\gamma^*\left[t_0^*,\infty\right)$ of perturbed problem (1), (2), (4)) crosses the switching set Φ_1.

Part 2. We will evaluate the Euclidean distance between the trajectory $\Gamma_1^*\left[t_0^*,\infty\right)$ and the point x_1. For this purpose, we introduce new function $T^*:\left[t_0,\infty\right)\to\left[t_0^*,\infty\right)$, defined in such way: For any $t\in\left[t_0,\infty\right)$, the corresponding function value $T^*=T^*(t)$ satisfies the restrictions:

$$\rho_E\left(X_1^*\left(T^*;t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right)=\rho_E\left(\Gamma_1^*\left[t_0^*,\infty\right),X_1\left(t;t_0,x_0\right)\right)$$

and

$$\rho_E\left(\Gamma_1^*\left[t_0^*,T^*\right),X_1\left(t;t_0,x_0\right)\right)<\rho_E\left(\Gamma_1^*\left[t_0^*,\infty\right),X_1\left(t;t_0,x_0\right)\right)$$
$$\Leftrightarrow\rho_E\left(X_1^*\left(T^*;t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right)<\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right),\;0\leq t^*<T^*=T^*(t).$$

In other words, for every point $t\geq t_0$, its corresponding moment $T^*\geq t_0^*$ is the first in which the distance between both points $X_1^*\left(T^*;t_0^*,x_0^*\right)$ and $X_1\left(t;t_0,x_0\right)$ is equal to the Euclidean distance between the trajectory $\Gamma_1^*\left[t_0^*,\infty\right)$ and the point $X_1\left(t;t_0,x_0\right)$ (fig. 2).

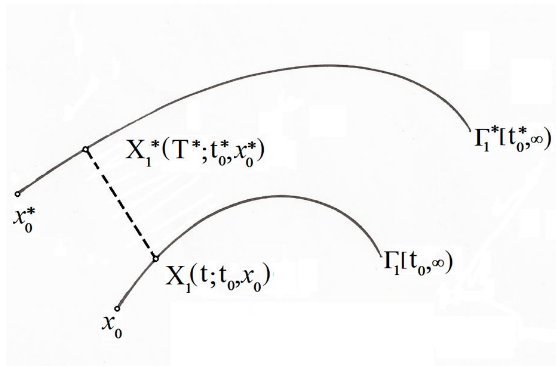

Fig. 2

As the system (1) (for $i=1$) is gravitating, then for any $t \ge t_0$, we have

$$\rho_E\left(X_1^*\left(T^*(t);t_0^*x_0^*\right),X_1(t;t_0,x_0)\right)=\rho_E\left(\Gamma_1^*\left[t_0^*,\infty\right),X_1(t;t_0,x_0)\right)$$

$$=\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right),X_1(t;t_0,x_0)\right),\ 0\le t^*<\infty\right\}$$

$$\le\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right),X_1(t;t_0,x_0)\right),\ 0\le t^*<\infty\right\},\ 0\le t<\infty\right\}\le\rho_H\left(\Gamma_1^*\left[t_0^*,\infty\right),\Gamma_1\left[t_0,\infty\right)\right)$$

$$\le\kappa_{j_1}.\rho_E\left(\Gamma_1^*\left[t_0^*,\infty\right),\Gamma_1\left[t_0,\infty\right)\right)\le\kappa_{j_1}.\rho_E\left(x_0^*,x_0\right)=k_{j_1}.\left\|x_0^*-x_0\right\|<\kappa_{j_1}.\delta=d<\varepsilon .$$

From the last inequality, we find that

$$\rho_E\left(\Gamma_1^*\left[t_0^*,\infty\right),x_1\right)=\rho_E\left(X_1^*\left(T^*(t_1);t_0^*,x_0^*\right),x_1\right)=\left\|X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)-x_1\right\|<d .$$

Part 3. We will find the estimate above of the distance

$$\rho_E\left(x_1^*,x_1\right)=\rho_E\left(x^*\left(t_1^*;t_0^*,x_0^*\right),x(t_1;t_0,x_0)\right).$$

Let point X_p^* be an orthogonal projection of point $X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)$ on the switching hyperplane Φ_1 (fig. 3).

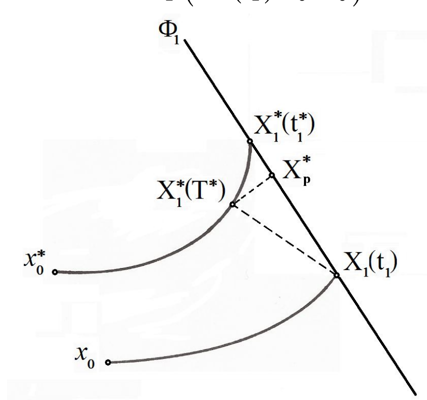

Fig. 3

The following notations are used in the figure:

$X_1(t_1)=X_1(t_1;t_0,x_0)$, $X_1^*\left(t_1^*\right)=X_1^*\left(t_1^*;t_0^*,x_0^*\right)$, $X_1^*\left(T^*\right)=X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)$. It is fulfilled

$$\left\|X_p^*-X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)\right\|\le\left\|x_1-X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)\right\|=\rho_E\left(X_1^*\left(T^*(t_1);t_0^*,x_0^*\right),x_1\right)\le d .$$

We have

$$\frac{\left|\left\langle x_1^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right),a_1\right\rangle\right|}{\left\|x_1^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)\right\|} = \frac{\left|\left\langle \int_{T^*(t_1)}^{t_1^*} f_1\left(X_1^*\left(\tau;t_0^*,x_0^*\right)\right)d\tau, a_1\right\rangle\right|}{\left\|\int_{T^*(t_1)}^{t_1^*} f_1\left(X_1^*\left(\tau;t_0^*,x_0^*\right)\right)d\tau\right\|}$$

$$= \frac{\left|\int_{T^*(t_1)}^{t_1^*}\left\langle f_1\left(X_1^*\left(\tau;t_0^*,x_0^*\right)\right),a_1\right\rangle d\tau\right|}{\left\|\int_{T^*(t_1)}^{t_1^*} f_1\left(X_1^*\left(\tau;t_0^*,x_0^*\right)\right)d\tau\right\|} \ge \frac{\left|\int_{T^*(t_1)}^{t_1^*}\mu C_{j_1}^2\left\|f_1\left(X_1^*\left(\tau;t_0^*,x_0^*\right)\right)\right\|d\tau\right|}{\left\|\int_{T^*(t_1)}^{t_1^*} f_1\left(X_1^*\left(\tau;t_0^*,x_0^*\right)\right)d\tau\right\|} = \mu C_{j_1}^2,$$

from where we find

$$\left\|x_1^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)\right\| \le \frac{1}{\mu.C_{j_1}^2}\left\langle x_1^* - X_p^* + X_p^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right),\ a_1\right\rangle$$

$$= \frac{1}{\mu.C_{j_1}^2}\left(\left\langle x_1^* - X_p^*,\ a_1\right\rangle + \left\langle X_p^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right),\ a_1\right\rangle\right) = \frac{1}{\mu.C_{j_1}^2}\left\|X_p^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)\right\| \le \frac{1}{\mu.C_{j_1}^2}.$$

By the last inequality, we conclude

$$\left\|x_1^* - X_p^*\right\| \le \left\|x_1^* - X_1^*\left(T^*(t_1);t_0^*,x_0^*\right)\right\| \le \frac{d}{\mu.C_{j_1}^2}.$$

Then

$$\rho_E\left(x_1^*,x_1\right) = \left\|x_1^* - x_1\right\| \le \left\|x_1^* - X_p^*\right\| + \left\|X_p^* - x_1\right\| \tag{13}$$

$$\le \left\|x_1^* - X_p^*\right\| + \left\|X_1^*\left(T^*(t_1);t_0^*,x_0^*\right) - x_1\right\| \le \frac{d}{\mu.C_{j_1}^2} + d = d\left(1 + \frac{1}{\mu.C_{j_1}^2}\right).$$

Part 4. We will find the estimate above of the distance between the points x_1^{*+} and x_1^{+}. The operator $\left(Id + I_1\right) = \left(Id + I_{j_1}\right)$ is a shrinking (see condition H7). Using condition H7 and inequality (13) we obtain the estimate

$$\rho_E\left(x_1^{*+},x_1^{+}\right) = \rho_E\left(x^*\left(t_1^*;t_0^*,x_0^*\right) + I_1\left(x^*\left(t_1^*;t_0^*,x_0^*\right)\right),\ x\left(t_1;t_0,x_0\right) + I_1\left(x\left(t_1;t_0,x_0\right)\right)\right) \tag{14}$$

$$\le C_{j_1}^4.\rho_E\left(x^*\left(t_1^*;t_0^*,x_0^*\right),x\left(t_1;t_0,x_0\right)\right) = C_{j_1}^4.\rho\left(x_1^*,x_1\right) \le d.C_{j_1}^4.\left(1 + \frac{1}{\mu.C_{j_1}^2}\right).$$

By condition 4 of Theorem 5, the inequalities

$$0 < \frac{\kappa_{j_1}C_{j_1}^4}{C_{j_1}^2\left(1 - \kappa_{j_1}C_{j_1}^4\right)} < 1$$

are valid. As μ is an arbitrary constant satisfying the inequality $0 < \mu < 1$, without loss of generality we assume $\frac{\kappa_{j_1}C_{j_1}^4}{C_{j_1}^2\left(1 - \kappa_{j_1}C_{j_1}^4\right)} \le \mu < 1$, from where we find that $C_{j_1}^4 \le \frac{\mu C_{j_1}^2}{\kappa_{j_1}\left(1 + \mu C_{j_1}^2\right)}$. From (14), using the last inequality, we find the estimate $\rho_E\left(x_1^{*+},x_1^{+}\right) \le d/\kappa_{j_1}$.

Part 5. Since d is an arbitrary constant satisfying the inequalities $0 \le d \le \min\{\varepsilon, r, \Delta\}$, we assume in additional that $d \le C_{j_1}^3$. Then by condition H6, it follows that for point x_1^+, it is valid

$$\rho_E\left(\Phi_1, x_1^+\right) = \rho_E\left(\Phi_1, x_1 + I_1(x_1)\right) \ge \rho_E\left(\Phi_1, (Id + I_1)(\Phi_1)\right)$$
$$= \rho_E\left(\Phi_{j_1}, \left(Id + I_{j_1}\right)\left(\Phi_{j_1}\right)\right) \ge C_{j_1}^3 \ge d .$$

The last inequality means that the point

$$x_1^+ \notin B\left(\Phi_1, d\right) = \left\{x \in D,\ \alpha_1 - d \le \langle a_1, x\rangle \le \alpha_1 + d\right\}.$$

Part 6. Let t_{-d} be a moment at which the following equalities are valid:
$\Gamma_1\left[t_0, \infty\right) \cap \Phi_{1-d} = X_1\left(t_{-d}; t_0, x_0\right)$ and $\Gamma_1\left[t_0, t_{-d}\right) \cap \Phi_{1-d} = \varnothing$.

(see Part 1). Clear that $0 < t_{-d} < t_1$. From Part 2, it follows that for every $t \ge t_0$, it is fulfilled

$$\rho_E\left(X_1^*\left(T^*(t); t_0^*, x_0^*\right), X_1\left(t; t_0, x_0\right)\right) < d .$$

Since the inequality $\rho_E\left(X_1\left(t; t_0, x_0\right), \Phi_1\right) \ge d$ is valid for $t_0 \le t \le t_{-d}$, then we deduce that

$$\rho_E\left(X_1^*\left(T^*(t); t_0^*, x_0^*\right), \Phi_1\right) \ge -\rho_E\left(X_1^*\left(T^*(t); t_0^*, x_0^*\right), X_1\left(t; t_0, x_0\right)\right) + \rho_E\left(X_1\left(t; t_0, x_0\right), \Phi_1\right) > -d + d = 0$$
$$\Leftrightarrow \Gamma_1^*\left[t_0^*, \infty\right) \cap \Phi_1 = \varnothing \Leftrightarrow T^*(t) < t_1^*,\ t_0 \le t \le t_{-d} .$$

Part 7. Using the results obtained in Part 6, we derive

$$\sup\left\{\inf\left\{\rho_E\left(x^*\left(t^*; t_0^*, x_0^*\right), x\left(t; t_0, x_0\right)\right),\ t_0^* \le t^* \le T^*\left(t_{-d}\right)\right\},\ t_0 \le t \le t_{-d}\right\}$$
$$= \sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*; t_0^*, x_0^*\right), X_1\left(t; t_0, x_0\right)\right),\ t_0^* \le t^* \le T^*\left(t_{-d}\right)\right\},\ t_0 \le t \le t_{-d}\right\}$$
$$= \sup\left\{\rho_E\left(\Gamma_1^*\left[t_0^*, T^*\left(t_{-d}\right)\right), X_1\left(t; t_0, x_0\right)\right),\ t_0 \le t \le t_{-d}\right\}$$
$$= \sup\left\{\rho_E\left(\Gamma_1^*\left[t_0^*, \infty\right), X_1\left(t; t_0, x_0\right)\right),\ t_0 \le t \le t_{-d}\right\} \le \rho_H\left(\Gamma_1^*\left[t_0^*, \infty\right), \Gamma_1\left[t_0, t_{-d}\right)\right)$$
$$\le \rho_H\left(\Gamma_1^*\left[t_0^*, \infty\right), \Gamma_1\left[t_0, \infty\right)\right) \le \kappa_{j_1}.\rho_E\left(\Gamma_1^*\left[t_0^*, \infty\right), \Gamma_1\left[t_0, \infty\right)\right) \le \kappa_{j_1}\left\|x_0^* - x_0\right\| \le \kappa_{j_1}\delta = d < \varepsilon .$$

Part 8. Here, we will evaluate above the difference $t_1 - t_{-d}$. Let point X_p be orthogonal projection of $X_1\left(t_{-d}; t_0, x_0\right)$ on the switching hyperplane Φ_1. It is satisfied

$$\left\|X_p - X_1\left(t_{-d}; t_0, x_0\right)\right\| = d .$$

We have

$$\frac{\left|\left\langle x_1 - X_1\left(t_{-d}; t_0, x_0\right), a_1\right\rangle\right|}{\left\|x_1 - X_1\left(t_{-d}; t_0, x_0\right)\right\|} = \frac{\left|\left\langle \int_{t_{-d}}^{t_1} f_1\left(X_1\left(\tau; t_0, x_0\right)\right) d\tau, a_1\right\rangle\right|}{\left\|\int_{t_{-d}}^{t_1} f_1\left(X_1\left(\tau; t_0, x_0\right)\right) d\tau\right\|}$$

$$= \frac{\left|\int_{t_{-d}}^{t_1} \left\langle f_1\left(X_1\left(\tau;t_0,x_0\right)\right), a\right\rangle d\tau\right|}{\left\|\int_{t_{-d}}^{t_1} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau\right\|} \geq \frac{\left|\int_{t_{-d}}^{t_1} \mu C_{j_1}^2 \left\|f_1\left(X_1\left(\tau;t_0,x_0\right)\right)\right\| d\tau\right|}{\left\|\int_{t_{-d}}^{t_1} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau\right\|} = \mu C_{j_1}^2,$$

From where, we find

$$\left\|x_1 - X_1\left(t_{-d};t_0,x_0\right)\right\| \leq \frac{1}{\mu.C_{j_1}^2}\left\langle x_1 - X_p + X_p - X_1\left(t_{-d};t_0,x_0\right), a_1\right\rangle$$

$$= \frac{1}{\mu.C_{j_1}^2}\left(\left\langle x_1 - X_p, a_1\right\rangle + \left\langle X_p - X_1\left(t_{-d};t_0,x_0\right), a_1\right\rangle\right) = \frac{1}{\mu.C_{j_1}^2}\left\|X_p - X_1\left(t_{-d};t_0,x_0\right)\right\| \leq \frac{d}{\mu.C_{j_1}^2}.$$

Function f_1 does not change its sign under $(t,x) \in R^+ \times B\left(\Phi_1, d\right)$ according to the second estimate of (7), we obtain

$$t_1 - t_{-d} = \int_{t_{-d}}^{t_1} d\tau \leq \frac{1}{C_{j_1}^6}\int_{t_{-d}}^{t_1} \left\|f_1\left(X_1\left(\tau;t_0,x_0\right)\right)\right\| d\tau = \frac{1}{C_{j_1}^6}\left\|\int_{t_{-d}}^{t_1} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau\right\|$$

$$= \frac{1}{C_{j_1}^6}\left\|X_1\left(t_1;t_0,x_0\right) - X_1\left(t_{-d};t_0,x_0\right)\right\| = \frac{1}{C_{j_1}^6}\left\|x_1 - X_1\left(t_{-d};t_0,x_0\right)\right\| \leq \frac{d}{\mu C_{j_1}^2 C_{j_1}^6}.$$

Part 9. Using the inequality from the previous part, we will evaluate the Euclidean distance

$$\rho_E\left(\Gamma_1\left[t_{-d},t_1\right], X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right)\right)$$

$$= \inf\left\{\rho_E\left(X_1\left(t;t_0,x_0\right), X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right)\right),\ t_{-d} \leq t \leq t_1\right\}$$

$$= \inf\left\{\rho_E\left(X_1\left(t_{-d};t_0,x_0\right) + \int_{t_{-d}}^{t} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau,\ X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right)\right),\ t_{-d} \leq t \leq t_1\right\}$$

$$= \inf\left\{\left\|X_1\left(t_{-d};t_0,x_0\right) + \int_{t_{-d}}^{t} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau - X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right)\right\|,\ t_{-d} \leq t \leq t_1\right\}$$

$$\leq \left\|X_1\left(t_{-d};t_0,x_0\right) - X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right)\right\| + \inf\left\{\left\|\int_{t_{-d}}^{t} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau\right\|,\ t_{-d} \leq t \leq t_1\right\}$$

$$\leq \rho_E\left(X_1\left(t_{-d};t_0,x_0\right), X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right)\right) + \inf\left\{\left\|\int_{t_{-d}}^{t} f_1\left(X_1\left(\tau;t_0,x_0\right)\right) d\tau\right\|,\ t_{-d} \leq t \leq t_1\right\}$$

$$\leq d + \frac{C_{j_1}^1 d}{\mu C_{j_1}^2 C_{j_1}^6} = \frac{\mu C_{j_1}^2 C_{j_1}^6 + C_{j_1}^1}{\mu C_{j_1}^2 C_{j_1}^6} d.$$

Part 10. We have

$$\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^* x_0^*\right), X_1\left(t;t_0,x_0\right)\right),\ t_0^* \leq t^* \leq t_1^*\right\},\ t_0 \leq t \leq t_1\right\} \qquad (15)$$

$$\leq \sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^* x_0^*\right), X_1\left(t;t_0,x_0\right)\right),\ t_0^* \leq t^* \leq T^*\left(t_{-d}\right) \leq t_1^*\right\},\ t_0 \leq t \leq t_1\right\}$$

$$\leq \max\left\{\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right), X_1\left(t;t_0,x_0\right)\right), t_1^* \leq t^* \leq T^*\left(t_{-d}\right)\right\}, t_0 \leq t \leq t_{-d}\right\},\right.$$

$$\left.\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right), X_1\left(t;t_0,x_0\right)\right), t^* = T^*\left(t_{-d}\right)\right\}, t_{-d} < t \leq t_1\right\}\right\}$$

$$\le \max\left\{d\,,\,\sup\left\{\rho_E\left(X_1^*\left(T^*\left(t_{-d}\right);t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right),\,t_{-d}<t\le t_1\right\}\right\}$$
$$=\frac{\mu C_{j_1}^2 C_{j_1}^6+C_{j_1}^1}{\mu C_{j_1}^2 C_{j_1}^6}d\,.$$

Analogously to the proof above, we conclude that

$$\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right),\,t_0\le t\le t_1\right\},\,t_0^*\le t^*\le t_1^*\right\}\le\frac{\mu C_{j_1}^2 C_{j_1}^6+C_{j_1}^1}{\mu C_{j_1}^2 C_{j_1}^6}d. \qquad (16)$$

Finally, from (15) and (16), it follows that

$$\rho_H\left\{\gamma^*\left[t_0^*,t_1^*\right],\gamma\left[t_0,t_1\right]\right\}=\rho_H\left\{\Gamma_1^*\left[t_0^*,t_1^*\right],\Gamma_1\left[t_0,t_1\right]\right\}$$
$$=\max\left\{\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right),\,t_0^*\le t^*\le t_1^*\right\},\,t_0\le t\le t_1\right\},\right.$$
$$\left.\sup\left\{\inf\left\{\rho_E\left(X_1^*\left(t^*;t_0^*,x_0^*\right),X_1\left(t;t_0,x_0\right)\right),\,t_0\le t\le t_1\right\},\,t_0^*\le t^*\le t_1^*\right\}\right\}$$
$$\le\frac{\mu C_{j_1}^2 C_{j_1}^6+C_{j_1}^1}{\mu C_{j_1}^2 C_{j_1}^6}d\,.$$

Part 11. From the previous parts of the proof, we derive the conclusion

$$\left(\forall\varepsilon=const>0\right)\left(\forall\mu=const,\,\frac{\kappa_j C_j^4}{C_j^2\left(1-\kappa_j C_j^4\right)}\le\mu<1;\;\;j=1,2,...,k\right)$$
$$\left(\exists\Delta=\Delta\left(\mu\right)>0\right):$$
$$\left(\forall d=const,\;0<d<\min\left\{\frac{\mu C_j^2 C_j^6\varepsilon}{\mu C_j^2 C_j^6+C_j^1},\Delta\left(\mu\right),C_j^3;\;\;j=1,2,...,k\right\}\right)$$
$$\left(\forall\delta=const,\,0<\delta<d/\kappa_j;\;j=1,2,...,k\right)\left(\forall t_0^*\in R^+,\;\left|t_0^*-t_0\right|<\delta\right)\left(\forall x_0^*\in D,\,\left\|x_0^*-x_0\right\|<\delta\right)\Rightarrow$$

11.1. $\rho_E\left(x_1^{*+},x_1^+\right)<d/\kappa_{j_1}$ (Part 4);

11.2. $x_1^+\notin B\left(\alpha,d\right)$ (Part 5);

11.3. $\lim_{i\to\infty}t_i=\infty$ (Theorem 2);

11.4. $\rho_H\left(\gamma^*\left[t_0^*,t_1^*\right],\gamma\left[t_0,t_1\right]\right)<\varepsilon$ (Part 10).

Part 12. Options are as follows:

12.1. The solution of the studied problem is subjected to a change of its structure and (of course) an impulsive effect taking place at the moment t_1 of the meeting between trajectory $\gamma\left[t_0,\infty\right)$ and switching set Φ_1. From 11.1 and 11.2, analogously to Theorem 4, it follows that

$$\rho_H\left(\gamma^*\left(t_1^*,\infty\right),\gamma\left(t_1,\infty\right)\right)=\rho_H\left(\gamma^*\left[t_1^*+0,\infty\right),\gamma\left[t_1+0,\infty\right)\right)=\rho_H\left(\Gamma_2^*\left[t_1^*+0,\infty\right),\Gamma_2\left[t_1+0,\infty\right)\right)<\varepsilon .$$

In this case, the proof follows from the last inequality and 11.4. Using Theorem 1, we obtain

$$\rho_H\left(\gamma^*\left[t_0^*,\infty\right),\gamma\left[t_0,\infty\right)\right)=\rho_H\left(\gamma^*\left[t_0^*,t_1^*,\right]\cup\gamma^*\left(t_1^*,\infty\right),\ \gamma\left[t_0,t_1\right]\cup\gamma\left(t_1,\infty\right)\right)$$
$$\le\max\left\{\rho_H\left(\gamma^*\left[t_0^*,t_1^*,\right],\gamma\left[t_0,t_1\right]\right),\ \rho_H\left(\gamma^*\left(t_1^*,\infty\right),\gamma\left(t_1,\infty\right)\right)\right\}<\varepsilon .$$

12.2. The solution of the considered problem is subjected to the finite number (e.g. - p) impulsive effects, accompanied by a change of the right hand side of the system. These changes take place at the switching moments $t_1,t_2,...,t_p$, in which trajectory $\gamma\left[t_0,\infty\right)$ meets the switching sets $\Phi_1,\Phi_2,...,\Phi_p$, respectively. Similarly to 11.1, 11.2 and 11.4, we find:

$$\rho_E\left(x_i^{*+},x_i^+\right)<d/\kappa_{j_i}\ ;\ \ x_i^+\notin B\left(\Phi_i,d\right);$$
$$\rho_H\left(\gamma^*\left(t_{i-1}^*,t_i^*\right],\gamma\left(t_{i-1},t_i\right]\right)<\varepsilon,\ \ i=1,2,...,p . \qquad (17)$$

Analogously to case 12.1 and using Theorem 4, we receive the inequality

$$\rho_H\left(\gamma^*\left[t_i^*,\infty\right),\gamma\left[t_i,\infty\right)\right)<\varepsilon .$$

In this case, the proof follows by (17), last inequality and Theorem 1.

12.3. Trajectory $\gamma\left[t_0,\infty\right)$ meets infinitely many times the switching sets $\Phi_1,\Phi_2,...$. Using induction, we obtain the following estimates:

$$\rho_H\left(\gamma^*\left(t_{i-1}^*,t_i^*\right],\gamma\left(t_{i-1},t_i\right]\right)<\varepsilon,\ \ i=1,2,....$$

Further, using Theorem 1, we find

$$\left(\forall p=1,2,...\right)\Rightarrow\ \rho_H\left(\gamma^*\left[t_0^*,t_p^*\right],\gamma\left[t_0,t_p\right]\right)\le\max\left\{\rho_H\left(\gamma^*\left(t_{i-1}^*,t_i^*\right],\gamma\left(t_{i-1},t_i\right]\right);\ i=1,2,...,p\right\}<\varepsilon .$$

Taking into account the fact that $\lim\limits_{p\to\infty}t_p=\infty$ (see Theorem 2), we deduce that

$$\rho_H\left(\gamma^*\left[t_i^*,\infty\right),\gamma\left[t_i,\infty\right)\right)=\lim_{p\to\infty}\ \rho_H\left(\gamma^*\left[t_0^*,t_p^*\right],\gamma\left[t_0,t_p\right]\right)\le\varepsilon .$$

The Theorem is proved.

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