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On New Integral Inequalities with Applications

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Abstract

We present several general integral inequalities for convex and concave mappings. Some new inequalities of the Simpson's type and the Hermite-Hadamard's type are obtained. Finally, some applications to special means of real numbers are also given.

Keywords: Convex functions, Concave functions, Inverse of Hölder's inequality, Favard's inequality, Bullen's inequality, Chebychev's inequality

1. Introduction

Let $f : I \subset R \to R$ be a convex function on the interval I of real numbers and $a,b \in I$ with $a<b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is known as Hermite-Hadamard's inequality for convex functions. Both inequalities hold in the reversed direction if f is concave. (Dragomir) [1]

In Mitrinović, [6, pp 64], the inverse of Hölder's inequality is given by the following theorem:

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Theorem A: Let functions $x \to f(x)^p$ and $x \to g(x)^q$ where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$ be positive and integrable on $[a,b]$ and let on $[a,b]$, $0<m_1 \le f(x) \le M_1 <+\infty$, $0<m_2 \le g(x) \le M_2 <+\infty$. Then

$$\left(\int_a^b f(x)^p dx\right)^{1/p}\left(\int_a^b g(x)^q dx\right)^{1/q} \le C_p \int_a^b f(x)g(x)dx$$

Where $C_p = \dfrac{M_1^p M_2^q - m_1^p m_2^q}{(pm_2M_2(M_1M_2^{q-1}-m_1m_2^{q-1}))^{1/p}(qm_1M_1(M_2M_1^{p-1}-m_2m_1^{p-1}))^{1/q}}$. (1)

In Latif, Pečarić & Peric, [5], Favard's inequality is given by the following theorem:

Theorem B: Let f be a concave nonnegative function on $[a,b]$. If $q>1$, then

$$\frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^q \ge \frac{1}{b-a}\int_a^b f(x)^q dx$$

If $0<q<1$, the reverse inequality holds.

In the literature, the following definition is well known:

Let $f:[a,b]\to R$ and $p \in R^+$. The p-norm of the function f on $[a,b]$ is defined by

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p dx\right)^{1/p}, & 0<p<\infty \\ \sup|f(x)|, & p=\infty \end{cases}$$

and $L^p[a,b]$ is the set of all functions $f:[a,b]\to R$ such that $\|f\|_p <\infty$.

For several recent results concerning Hermite-Hadamard inequality and double integral inequalities, we refer the reader to Kirmaci [3-4].

In this paper, we derive several general integral identities for convex and concave mappings. Some new Simpson's type , midpoint type and trapezoid type inequalities are written. Further, some applications for special means of real numbers are provided.

2. Main Results

Firstly, we start by the following lemma:

Lemma: Let $f : I \subset R \to R$ be twice differentiable mapping on I^0 such that $f'' \in L[a,b]$ and $0 \le \lambda \le 1$, where $a,b \in I$ with $a < b$. Then we have the equality

$$\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} (x-a)\left(\frac{a+b}{2}-x-\lambda\right)f''(x)dx+\int_{\frac{a+b}{2}}^{b} (b-x)\left(x-\frac{a+b}{2}-\lambda\right)f''(x)dx\right]=$$

$$=\Delta(x,\lambda)=\left(\frac{\lambda}{b-a}+\frac{1}{2}\right)f\left(\frac{a+b}{2}\right)+\left(\frac{1}{2}-\frac{\lambda}{b-a}\right)\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \quad (2)$$

Proof: By integration by parts twice, we obtain

$$I_1=\int_a^{\frac{a+b}{2}} (x-a)\left(\frac{a+b}{2}-x-\lambda\right)f''(x)dx=$$

$$=-\frac{b-a}{2}\lambda f'\left(\frac{a+b}{2}\right)+\left(\lambda+\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)+\left(\frac{b-a}{2}-\lambda\right)f(a)-2\int_a^{\frac{a+b}{2}} f(x)dx$$

and

$$I_2=\int_{\frac{a+b}{2}}^{b} (b-x)\left(x-\frac{a+b}{2}-\lambda\right)f''(x)dx=$$

$$= \frac{b-a}{2}\lambda f'\left(\frac{a+b}{2}\right) + \left(\lambda + \frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2} - \lambda\right) f(b) - 2\int_{\frac{a+b}{2}}^{b} f(x)dx$$

By adding these equalities side by side and by multiplying both sides by $1/2(b-a)$, we obtain equality (2).

Remark : Taking $\lambda = -\frac{b-a}{2}$ in (2), we obtain the identity

$$\frac{1}{2}\int_a^b (x-a)(b-x)f''(x)dx = (b-a)\left(\frac{f(a)+f(b)}{2}\right) - \int_a^b f(x)dx$$

which may be found in Dragomir, Cerone & Sofo([1, pp. 38] and [2]).

Therefore, we can state the following results:

Theorem 1: Let $f : I \subset R \to R$ be twice differentiable mapping on I^0 such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$. If the mapping

$$\varphi(x) = \begin{cases} (x-a)\left(\frac{a+b}{2} - x - \lambda\right) f''(x) & , x \in \left[a, \frac{a+b}{2}\right) \\ (b-x)\left(x - \frac{a+b}{2} - \lambda\right) f''(x) & , x \in \left[\frac{a+b}{2}, b\right] \end{cases}$$

is convex on $[a,b]$, then we have the inequality, for $0 \le \lambda \le 1$

$$\frac{b-a}{16}\left(\frac{b-a}{4} - \lambda\right)\left[f''\left(\frac{3a+b}{4}\right) + f''\left(\frac{a+3b}{4}\right)\right] \le \Delta(x,\lambda) \le$$

$$\le \frac{b-a}{32}\left(\frac{b-a}{4} - \lambda\right)\left[f''\left(\frac{3a+b}{4}\right) + f''\left(\frac{a+3b}{4}\right)\right] - \frac{b-a}{16}\lambda f''\left(\frac{a+b}{2}\right) \qquad (3)$$

Proof: Applying the first inequality of Hermite-Hadamard for the mapping φ we write

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx \geq \varphi\left(\frac{3a+b}{4}\right)=\frac{b-a}{4}\left(\frac{b-a}{4}-\lambda\right)f''\left(\frac{3a+b}{4}\right)$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)dx \geq \varphi\left(\frac{a+3b}{4}\right)=\frac{b-a}{4}\left(\frac{b-a}{4}-\lambda\right)f''\left(\frac{a+3b.}{4}\right)$$

Applying the Bullen's inequality for the mapping φ, we have

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx \leq \frac{1}{2}\left[\varphi\left(\frac{3a+b}{4}\right)+\frac{\varphi(a)+\varphi\left(\frac{a+b}{2}\right)}{2}\right]=\frac{b-a}{8}\left(\frac{b-a}{4}-\lambda\right)f''\left(\frac{3a+b}{4}\right)-\frac{b-a}{8}\lambda f''\left(\frac{a+b}{2}\right)$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)dx \leq \frac{1}{2}\left[\varphi\left(\frac{a+3b}{4}\right)+\frac{\varphi\left(\frac{a+b}{2}\right)+\varphi(b)}{2}\right]=\frac{b-a}{8}\left(\frac{b-a}{4}-\lambda\right)f''\left(\frac{a+3b.}{4}\right)-\frac{b-a}{8}\lambda f''\left(\frac{a+b}{2}\right)$$

Adding all these inequalities and from (2), we obtain inequality (3). Thus the proof is completed.

Theorem 2: Let $f: I \subset R \to R$ be twice differentiable mapping on I^o. If $f'' \in L[a,b]$ and f'' is a positive mapping for all $x \in [a,b] \subset I^o$ and the mapping $\varphi(x)$ is convex on $[a,b]$, then we have the inequality, for $0 \leq \lambda \leq 1$

$$\frac{b-a}{16}\left(\frac{b-a}{4}-\lambda\right)\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]\le\Delta(x,\lambda)\le\frac{1}{2(b-a)}\left(\frac{\alpha_1\alpha_2}{\alpha_3}+\frac{\alpha_4\alpha_5}{\alpha_6}\right) \quad (4)$$

where $\alpha_i,(i=1,...,6)$ are given by (7)-(12) respectively.

Proof: By Chebychev integral inequality for asynchronous mappings, we have the following inequalities:

$$\Gamma_1=\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x-\lambda\right)f''(x)dx\le\frac{1}{2(b-a)}\frac{\int_a^{\frac{a+b}{2}}(x-a)f''(x)dx\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x-\lambda\right)f''(x)dx}{\int_a^{\frac{a+b}{2}}f''(x)dx} \quad (5)$$

$$\Gamma_2=\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b}(b-x)\left(x-\frac{a+b}{2}-\lambda\right)f''(x)dx\le\frac{1}{2(b-a)}\frac{\int_{\frac{a+b}{2}}^{b}(b-x)f''(x)dx\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}-\lambda\right)f''(x)dx}{\int_{\frac{a+b}{2}}^{b}f''(x)dx} \quad (6)$$

By integration by parts, we have the following equalities:

$$\int_a^{\frac{a+b}{2}}(x-a)f''(x)dx=\left(\frac{b-a}{2}\right)f'\left(\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)+f(a)=\alpha_1 \quad (7)$$

$$\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x-\lambda\right)f''(x)dx=(\lambda-\left(\frac{b-a}{2}\right))f'(a)-\lambda f'(\frac{a+b}{2})+f\left(\frac{a+b}{2}\right)-f(a)=\alpha_2 \quad (8)$$

$$\int_a^{\frac{a+b}{2}}f''(x)dx=f'\left(\frac{a+b}{2}\right)-f'(a)=\alpha_3 \quad (9)$$

$$\int_{\frac{a+b}{2}}^{b}(b-x)f''(x)dx=-\left(\frac{b-a}{2}\right)f'(\frac{a+b}{2})-f\left(\frac{a+b}{2}\right)+f(b)=\alpha_4 \quad (10)$$

$$\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}-\lambda\right)f''(x)dx=(\left(\frac{b-a}{2}\right)-\lambda)f'(b)+\lambda f'(\frac{a+b}{2})+f\left(\frac{a+b}{2}\right)-f(b)=\alpha_5 \quad (11)$$

$$\int_{\frac{a+b}{2}}^{b} f''(x)dx = f'(b)-f'\left(\frac{a+b}{2}\right)=\alpha_6 \quad (12)$$

Substituting equalities (7),(8),(9) in inequality (5), we have

$$I'_1 \le \frac{1}{2(b-a)}\left(\frac{\alpha_1\alpha_2}{\alpha_3}\right) \quad (13)$$

Substituting equalities (10),(11),(12) in inequality (6), we have

$$I'_2 \le \frac{1}{2(b-a)}\left(\frac{\alpha_4\alpha_5}{\alpha_6}\right) \quad (14)$$

Adding (13) and (14), we obtain

$$\Delta(x,\lambda)=I'_1+I'_2 \le \frac{1}{2(b-a)}\left(\frac{\alpha_1\alpha_2}{\alpha_3}+\frac{\alpha_4\alpha_5}{\alpha_6}\right) \quad (15)$$

By (15) and using left hand side of double inequality (3), we deduce double inequality (4).

This concludes the proof.

Theorem 3: Let $f: I\subset R\to R$ be twice differentiable mapping on I^0. Let $|f''|$ be q-Lebesque integrable on $[a,b]$, where $a,b\in I$ with $a<b$. Let on $[a,\frac{a+b}{2})$, mappings $g_1(x)^q$ and $f''(x)^q$ and on $[\frac{a+b}{2},b]$, mappings $g_2(x)^q$ and $f''(x)^q$ be monotone in the same sense and integrable. If the mapping $\varphi(x)$ is concave nonnegative on $[a,b]$, then we have the inequality, for $q>1$ and $0\le\lambda\le 1$

$$\frac{(q+1)^{1/q}2^{1/q}}{32(b-a)^{\frac{1}{q}-2}}(1-\lambda_1)^{2+\frac{1}{q}}\left[B(q+1,q+1,\frac{1}{1-\lambda_1})\right]^{1/q}\|f''\|_q \le \Delta(x,\lambda) \le$$

$$\le \frac{b-a}{16}\left(\frac{b-a}{4}-\lambda\right)\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right] \quad (16)$$

where $f''(x)^q = (f''(x))^q$ and

$$g_1(x) = (x-a)(\frac{a+b}{2}-x-\lambda), g_2(x) = (b-x)(x-\frac{a+b}{2}-\lambda).$$

Proof: Applying the second inequality of Hermite-Hadamard for the concave nonnegative mapping φ, we write

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx \le \varphi\left(\frac{3a+b}{4}\right) = \frac{b-a}{4}\left(\frac{b-a}{4}-\lambda\right)f''\left(\frac{3a+b}{4}\right)$$

And

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)dx \le \varphi\left(\frac{a+3b}{4}\right) = \frac{b-a}{4}\left(\frac{b-a}{4}-\lambda\right)f''\left(\frac{a+3b.}{4}\right)$$

Adding these inequalities and from (2), we obtain

$$\frac{1}{2(b-a)}\int_a^b\varphi(x)dx \le \frac{b-a}{16}\left(\frac{b-a}{4}-\lambda\right)\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right] \quad (17)$$

Also, applying the Favard's inequality for the mapping φ, we obtain for $q>1$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx \ge \frac{(q+1)^{1/q}}{2}(\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)^q dx)^{1/q}$$

And

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)dx \geq \frac{(q+1)^{1/q}}{2}\left(\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)^{q}dx\right)^{1/q}$$

Adding these inequalities and from (2), we obtain

$$\frac{1}{2(b-a)}\int_{a}^{b}\varphi(x)dx \geq$$

$$\geq \frac{(q+1)^{1/q}2^{1/q}}{8(b-a)^{1/q}}\left[\left(\int_{a}^{\frac{a+b}{2}}(x-a)^{q}\left(\frac{a+b}{2}-x-\lambda\right)^{q}f''(x)^{q}dx\right)^{1/q}+\left(\int_{\frac{a+b}{2}}^{b}(b-x)^{q}\left(x-\frac{a+b}{2}-\lambda\right)^{q}f''(x)^{q}dx\right)^{1/q}\right]$$

By Chebyshev's inequality, we deduce

$$\frac{1}{2(b-a)}\int_{a}^{b}\varphi(x)dx \geq \frac{(q+1)^{1/q}2^{1/q}}{8(b-a)^{1/q}}\left(\frac{2}{b-a}\right)^{1/q}\left[\left(\int_{a}^{\frac{a+b}{2}}(x-a)^{q}\left(\frac{a+b}{2}-x-\lambda\right)^{q}dx\right)^{1/q}\left(\int_{a}^{\frac{a+b}{2}}f''(x)^{q}dx\right)^{1/q}+\right.$$

$$\left.+\left(\int_{\frac{a+b}{2}}^{b}(b-x)^{q}\left(x-\frac{a+b}{2}-\lambda\right)^{q}dx\right)^{1/q}\left(\int_{\frac{a+b}{2}}^{b}(f''(x)^{q}dx\right)^{1/q}\right] \tag{18}$$

Using the change of the variable $x=(1-t)a+t\left(\frac{a+b}{2}\right)$ and from $dx=\left(\frac{b-a}{2}\right)dt$, we write

$$\int_{a}^{\frac{a+b}{2}}(x-a)^{p}\left(\frac{a+b}{2}-x-\lambda\right)^{p}dx=\left(\frac{b-a}{2}\right)\int_{0}^{1}\left((1-t)a+t\left(\frac{a+b}{2}\right)-a\right)^{p}\left(\frac{a+b}{2}-(1-t)a-t\left(\frac{a+b}{2}\right)-\lambda\right)^{p}dt$$

$$=\left(\frac{b-a}{2}\right)^{2p+1}\int_0^1 t^p(1-t-\lambda_1)^p dt=\left(\frac{b-a}{2}\right)^{2p+1}(1-\lambda_1)^{2p+1}\int_0^{1/(1-\lambda_1)} t_1{}^p(1-t_1)^p dt_1$$

$$=\left(\frac{b-a}{2}\right)^{2p+1}(1-\lambda_1)^{2p+1}B(p+1,p+1,\frac{1}{1-\lambda_1}) \qquad (19)$$

where, $\lambda_1=\frac{\lambda}{(b-a)/2}, t_1=\frac{t}{1-\lambda_1}, dt=(1-\lambda_1)dt_1$, $B(p,q,x)=\int_0^x t^{p-1}(1-t)^{q-1}dt$, $(p,q>0)$.

Similarly, using the change of the variable $x=(1-t)b+t(\frac{a+b}{2})$ and from

$dx=\left(\frac{a-b}{2}\right)dt$, we have

$$\int_{\frac{a+b}{2}}^{b}(b-x)^p\left(x-\frac{a+b}{2}-\lambda\right)^p dx=\left(\frac{b-a}{2}\right)\int_0^1\left(b-(1-t)b-t(\frac{a+b}{2})\right)^p\left((1-t)b+t(\frac{a+b}{2})-\frac{a+b}{2}-\lambda\right)^p dt$$

$$=\left(\frac{b-a}{2}\right)^{2p+1}\int_0^1 t^p(1-t-\lambda_1)^p dt=\left(\frac{b-a}{2}\right)^{2p+1}(1-\lambda_1)^{2p+1}B(p+1,p+1,\frac{1}{1-\lambda_1}) \qquad (20)$$

Let $a_1=\int_a^{\frac{a+b}{2}}\left|f''(x)\right|^q dx, b_1=\int_{\frac{a+b}{2}}^{b}\left|f''(x)\right|^q dx$. Here $0<1/q<1$, for q>1. Using the fact

that,

$$\sum_{k=1}^{n}(a_k+b_k)^s\le\sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s$$

for $0\le s<1$, we obtain,

$$\|f''\|_q=(\int_a^b|f''(x)|^q dx)^{1/q}\le(\int_a^{\frac{a+b}{2}}|f''(x)|^q dx)^{1/q}+(\int_{\frac{a+b}{2}}^b|f''(x)|^q dx)^{1/q}=\|f''\|_{q,a}+\|f''\|_{q,b} \quad (21)$$

Substituting equalities (19),(20) and inequality (21) in inequality (18), we have

$$\frac{1}{2(b-a)}\int_a^b\varphi(x)dx$$

$$\ge\frac{(q+1)^{1/q}2^{1/q}}{32(b-a)^{\frac{1}{q}-2}}(1-\lambda_1)^{2+\frac{1}{q}}\left[B(q+1,q+1,\frac{1}{1-\lambda_1})\right]^{1/q}(\|f''\|_{q,a}+\|f''\|_{q,b})$$

$$\ge\frac{(q+1)^{1/q}2^{1/q}}{32(b-a)^{\frac{1}{q}-2}}(1-\lambda_1)^{2+\frac{1}{q}}\left[B(q+1,q+1,\frac{1}{1-\lambda_1})\right]^{1/q}\|f''\|_q \quad (22)$$

From (17) and (22), we deduce inequality (16). This concludes the proof.

Theorem 4: Let $f:I\subset R\to R$ be twice differentiable mapping on I^0. Let $|f''|$ be q-Lebesque integrable on $[a,b]$, where $a,b\in I$ with $a<b$. Let on $[a,\frac{a+b}{2})$, mappings $g_1(x)^q$ and $f''(x)^q$ and on $[\frac{a+b}{2},b]$, mappings $g_2(x)^q$ and $f''(x)^q$ be monotone in the opposite sense and integrable. If the mapping $\varphi(x)$ is concave nonnegative on $[a,b]$, then we have the inequality, for $0<q<1$ and $0\le\lambda\le1$

$$-\frac{b-a}{8}\lambda f''\left(\frac{a+b}{2}\right)\le\Delta(x,\lambda)\le\frac{(q+1)^{1/q}2^{1/q}}{32(b-a)^{\frac{1}{q}-2}}(1-\lambda_1)^{2+\frac{1}{q}}\left[B(q+1,q+1,\frac{1}{1-\lambda_1})\right]^{1/q}(\|f''\|_{q,a}+\|f''\|_{q,b}) \quad (23)$$

where $\|f''\|_{q,a},\|f''\|_{q,b}$ are given by (21), $g_1(x),g_2(x)$ are as in Theorem 3 and $f''(x)^q=(f''(x))^q$.

Proof: Applying the first inequality of Hermite-Hadamard for the concave nonnegative mapping φ, we obtain inequalities

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx \geq \frac{\varphi(a)+\varphi\left(\frac{a+b}{2}\right)}{2} = -\frac{b-a}{4}\lambda f''\left(\frac{a+b}{2}\right)$$

And

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)dx \geq \frac{\varphi\left(\frac{a+b}{2}\right)+\varphi(b)}{2} = -\frac{b-a}{4}\lambda f''\left(\frac{a+b}{2}\right)$$

Adding these inequalities and from (2), we obtain

$$\frac{1}{2(b-a)}\int_a^b\varphi(x)dx \geq -\frac{b-a}{8}\lambda f''\left(\frac{a+b}{2}\right) \tag{24}$$

Also, applying the Favard's inequality for the mapping φ, we obtain for $0<q<1$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx \leq \frac{(q+1)^{1/q}}{2}\left(\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)^q dx\right)^{1/q}$$

And

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)dx \leq \frac{(q+1)^{1/q}}{2}\left(\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\varphi(x)^q dx\right)^{1/q}$$

Adding these inequalities and from equality (2), we obtain

$$\frac{1}{2(b-a)}\int_a^b\varphi(x)dx \leq$$

$$\leq \frac{(q+1)^{1/q}2^{1/q}}{8(b-a)^{1/q}}\left[\left(\int_a^{\frac{a+b}{2}}(x-a)^q\left(\frac{a+b}{2}-x-\lambda\right)^q f''(x)^q dx\right)^{1/q} + \left(\int_{\frac{a+b}{2}}^{b}(b-x)^q\left(x-\frac{a+b}{2}-\lambda\right)^q f''(x)^q dx\right)^{1/q}\right]$$

Since on $[a,\frac{a+b}{2})$, mappings g_1^q and f''^q and on $[\frac{a+b}{2},b]$, mappings g_2^q and f''^q are monotone in the opposite sense and integrable, we may apply Chebyshev's inequality. Hence, we write

$$\frac{1}{2(b-a)}\int_a^b \varphi(x)dx \le \frac{(q+1)^{1/q}2^{1/q}}{8(b-a)^{1/q}}\frac{2^{1/q}}{(b-a)^{1/q}}\Bigg[(\int_a^{\frac{a+b}{2}} (x-a)^q(\frac{a+b}{2}-x-\lambda)^q dx)^{1/q} (\int_a^{\frac{a+b}{2}} f''(x)^q dx)^{1/q} +$$

$$+(\int_{\frac{a+b}{2}}^b (b-x)^q(x-\frac{a+b}{2}-\lambda)^q dx)^{1/q} (\int_{\frac{a+b}{2}}^b (f''(x)^q dx)^{1/q}\Bigg]$$

Using equalities (19) and (20), we obtain

$$\frac{1}{2(b-a)}\int_a^b \varphi(x)dx \le \frac{(q+1)^{1/q}2^{1/q}}{32(b-a)^{\frac{1}{q}-2}}(1-\lambda_1)^{2+\frac{1}{q}}\left[B(q+1,q+1,\frac{1}{1-\lambda_1})\right]^{1/q}(\|f''\|_{q,a}+\|f''\|_{q,b}) \tag{25}$$

From (24) and (25), we get inequality (23). This concludes the proof.

3. Applications to Quadrature Formulas

From the double inequalities in Section 2, we obtain the following new inequalities of the midpoint type, trapezoid type and Simpson's type:

Proposition 1: With the assumptions Theorem 1,

(i). If $\lambda = -\frac{b-a}{2}$, we have trapezoid inequality

$$\frac{3(b-a)^2}{64}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right] \le \left[\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx \le$$

$$\le \frac{3(b-a)^2}{128}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]+\frac{(b-a)^2}{32}f''\left(\frac{a+b}{2}\right)$$

(ii). If $\lambda = \frac{b-a}{2}$, we have midpoint inequality

$$-\frac{(b-a)^2}{64}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right] \le f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \le$$

$$\le -\frac{(b-a)^2}{128}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]-\frac{(b-a)^2}{32}f''\left(\frac{a+b}{2}\right)$$

(iii). If $\lambda = \frac{b-a}{4}$, we have simpson inequality

$$0 \le \left[\frac{f(a)+f(b)}{6}\right]+\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \le -\frac{(b-a)^2}{64}f''\left(\frac{a+b}{2}\right)$$

Proposition 2: With the assumptions Theorem 2,

(i). If $\lambda = \frac{b-a}{4}$, we have simpson inequality

$$0 \le \left[\frac{f(a)+f(b)}{6}\right]+\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2(b-a)}\left(\frac{\alpha_1\alpha_2}{\alpha_3}+\frac{\alpha_4\alpha_5}{\alpha_6}\right)$$

where, $\alpha_2 = -\left(\frac{b-a}{4}\right)f'(a)-\frac{b-a}{4}f'(\frac{a+b}{2})+f\left(\frac{a+b}{2}\right)-f(a)$,

$\alpha_5 = \left(\frac{b-a}{4}\right)f'(b)+\frac{b-a}{4}f'(\frac{a+b}{2})+f\left(\frac{a+b}{2}\right)-f(b)$ and $\alpha_1,\alpha_3,\alpha_4,\alpha_6$ are given by (7,9,10,12) respectively.

(ii). If $\lambda = \frac{b-a}{2}$, we have midpoint inequality

$$-\frac{(b-a)^2}{64}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right] \le f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \le$$

$$\leq \frac{1}{2(b-a)}\left(\frac{\alpha_1\alpha_2}{\alpha_3}+\frac{\alpha_4\alpha_5}{\alpha_6}\right)$$

where, $\alpha_2 = -\frac{b-a}{2} f'(\frac{a+b}{2}) + f\left(\frac{a+b}{2}\right) - f(a)$,

$$\alpha_5 = \frac{b-a}{2} f'(\frac{a+b}{2}) + f\left(\frac{a+b}{2}\right) - f(b)$$

and $\alpha_1,\alpha_3,\alpha_4,\alpha_6$ are given by (7,9,10,12) respectively.

Proposition 3: With the assumptions Theorem 3, for $q>1$

(i). If $\lambda = -\frac{b-a}{2}$, we have trapezoid inequality

$$\frac{(q+1)^{1/q}(b-a)^{2-1/q}4^{\frac{1}{q}}}{8}\left[B(q+1,q+1,\frac{1}{2})\right]^{1/q}\|f''\|_q \leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \leq$$

$$\leq \frac{3(b-a)^2}{64}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]$$

(ii). If $\lambda = \frac{b-a}{4}$, we have simpson inequality

$$\frac{(q+1)^{1/q}(b-a)^{2-1/q}}{128}\left[B(q+1,q+1,2)\right]^{1/q}\|f''\|_q \leq$$

$$\leq \left[\frac{f(a)+f(b)}{6}\right]+\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx \leq 0$$

Proposition 4: With the assumptions Theorem 4, for $0<q<1$

(i). If $\lambda = \frac{b-a}{4}$, we have simpson inequality

$$-\frac{(b-a)^2}{32} f''\left(\frac{a+b}{2}\right) \le \left[\frac{f(a)+f(b)}{6}\right] + \frac{2}{3} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \le$$

$$\le \frac{(q+1)^{1/q}(b-a)^{2-1/q}}{128}\left[B(q+1,q+1,2)\right]^{1/q}(\|f''\|_{q,a} + \|f''\|_{q,b})$$

(ii). If $\lambda = -\frac{b-a}{2}$, we have trapezoid inequality

$$\frac{(b-a)^2}{16} f''\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \le$$

$$\le \frac{(q+1)^{1/q}(b-a)^{2-1/q} 4^{\frac{1}{q}}}{8}\left[B(q+1,q+1,\frac{1}{2})\right]^{1/q}(\|f''\|_{q,a} + \|f''\|_{q,b})$$

4. Applications To Special Means

we shall consider the means for arbitrary real numbers $\alpha,\beta,\alpha\neq\beta$. We take

$A(\alpha,\beta) = \frac{\alpha+\beta}{2}$, $\alpha,\beta\in R$, (arithmetic mean)

$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}$, $n\in Z\setminus\{-1,0\}$, $\alpha,\beta\in R$, $\alpha\neq\beta$, (generalized log-mean)

For the results in Sections 2-3, we give some applications to special means of real numbers.

Proposition 5: Let $0 = a < b$. Then we have the inequality, for $0 \le \lambda \le 1$

$$\frac{b-a}{8}\left(\frac{b-a}{4}-\lambda\right)\left[A\left(-\frac{3a+b}{4}, -\frac{a+3b}{4}\right)\right] \le$$

$$\le -\frac{1}{6}\left[\left(\frac{\lambda}{b-a}+\frac{1}{2}\right)A^3(a,b) + \left(\frac{1}{2}-\frac{\lambda}{b-a}\right)A\left(a^3,b^3\right)\right] + \frac{1}{6}L_3^3(a,b) \le$$

$$\leq \frac{b-a}{16}(\frac{b-a}{4}-\lambda)\left[A\left(-\frac{3a+b}{4},-\frac{a+3b}{4}\right)\right]-\frac{b-a}{16}\lambda A(-a,-b)$$

Proof: The assertion follows from Theorem 1 applied for $f(x)=-\frac{1}{6}x^3$.

Proposition 6: Let $0=a<b$. Then we have the inequality

$$-\frac{3(b-a)^2(a+b)}{64}\leq -\frac{1}{6}A(a^3,b^3)+\frac{1}{6}L_3^3(a,b)\leq -\frac{5(b-a)^2(a+b)}{128}$$

Proof: The assertion follows from Proposition 1-i applied for $f(x)=-\frac{1}{6}x^3$.

Proposition 7: Let $0<a<b$. Then we have the inequality, for $q>1$

$$\frac{(q+1)^{1/q}(b-a)^2}{64}\left[B(q+1,q+1,2)\right]^{1/q}\leq \frac{1}{3}A(a^2,b^2)+\frac{2}{3}A^2(a,b)-L_2^2(a,b)\leq 0$$

Proof: The assertion follows from Proposition 3-ii applied for $f(x)=x^2$.

Proposition 8: Let $0<a<b$. Then we have the inequality, for $0<q<1$

$$-\frac{(b-a)^2}{16}\leq \frac{1}{3}A(a^2,b^2)+\frac{2}{3}A^2(a,b)-L_2^2(a,b)\leq$$
$$\leq \frac{(q+1)^{1/q}(b-a)^2}{32(2)^{1/q}}\left[B(q+1,q+1,2)\right]^{1/q}$$

Proof: The assertion follows from Proposition 4-i applied for $f(x)=x^2$.

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