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Finite – Time Ruin Probability for Sequences of Dependent Random Variables with Interest

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Abstract

The aim of this paper is to build an exact formula for ruin probability of generalized risk processes under interest force with assumption that claims and premiums are assumed to be positive-valued random variables and interests are assumed to be non - negative- valued random variables (claims, premiums and interests are assumed to be independent). In addition, they are homogeneous Markov chains. This situation is quite realistic for many situations. An exact formula for ruin (non-ruin) probabilities is derived in this paper.

Keywords: Ruin probability, Non- Ruin probability, Homogeneous Markov chain

1. Introduction

In the risk theory, there has been a major interest in actuarial science. Since a large portion of the surplus of insurance business from investment income, actuaries have been studying ruin problems under risk models with rates of interest. For example, Teugels and Sundt [9], [10] studied the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang [12] established both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Cai [1], [2] investigated the ruin probabilities in two risk models, with independent premiums and claims and used a first-order autoregressive process to model the rates of in interest.

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Cai and Dickson [3] obtained Lundberg inequalities for ruin probabilities in two discrete-time risk process with a Markov chain interest model and independent premiums and claims. Promislow, S., D. [5], given upper bounds for ruin in a process with dependent increments. Xu, L. and Wang, R. [11] established both exponential and non – exponential upper bounds for ruin probabilities in a autoregressive risk model with Markov chain interest rate. However, those results is only given upper bounds for finite-time probabilities and ultimate ruin probability that they did not provide an exact formula for finite-time probabilities.

Claude Lefèvre and Stéphane Loisel [4] studied the problem of ruin in the classical compound binomial and compound Poisson risk models. Their primary purpose is to extend those models which is an exact formula derived by Pircard and Lefèvre [4] for the probability of (non-ruin) ruin within finite time.

However, Claude Lefèvre and Stéphane Loisel [4] did not provide an exact formula for ruin probability of generalized risk processes under interest force with surplus process $\{U_t\}_{t\geq 1}$ of insurance company written as

$$U_t = U_{t-1}(1+I_t) + X_t - Y_t; t = 1,2,.. \quad (1.1)$$

or

$$U_t = (U_{t-1} + X_t)(1+I_t) - Y_t; t = 1,2,... \quad (1.2)$$

where $U_o = u$ is initial surplus, u and t are positive integer numbers, $X = \{X_i\}_{i\geq 1}$ are premiums of the company, $Y = \{Y_j\}_{j\geq 1}$ are claim of the company , X_i and Y_i take values in a finite set of positive numbers; $I = \{I_k\}_{k\geq 1}$ are interests of company, I_i take values in a finite set of non – negative numbers. X , Y and I are assumed to be independent.

In [7], Nguyen Thi Thuy Hong built an exact formula for ruin (non-ruin) probability for model:

$$U_t = u + \sum_{i=1}^{t} X_i - \sum_{i=1}^{t} Y_i \qquad (1.3)$$

with u, t, X_i, Y_i are positive integer number.

Phung Duy Quang [8] extended the result of Nguyen Thi Thuy Hong, the author built an exact formula for ruin (non-ruin) probability for model:

$$U_t = u(1+r)^t + \sum_{i=1}^{t} X_i (1+r)^{t-i+1} - \sum_{i=1}^{t} Y_i (1+r)^{t-1} \qquad (1.4)$$

with u, t, X_i, Y_i are positive integer number, r is positive constant interest.

In [6], Bui Khoi Dam and Phung Duy Quang built an exact formula for ruin (non-ruin) probability for model (1.1) and (1.2) with $X = \{X_i\}_{i\geq 1}$ and $Y = \{Y_j\}_{j\geq 1}$ are independent identically or non identically distributed positive-valued random variables; $I = \{I_k\}_{k\geq 1}$ are independent identically or non identically distributed non-negative-valued random variables. In addition, X, Y and I are assumed to be independent.

The aim of this paper is to build an exact formula for finte time ruin (non-ruin) probability of model (1.1) and (1.2) with $X = \{X_i\}_{i\geq 1}$, $Y = \{Y_j\}_{j\geq 1}$ and $I = \{I_k\}_{k\geq 1}$ are homogeneous Markov chains. In addition, X, Y and I are assumed to be independent. This result also extends the ruin probability to the general model, which is given in [6].

2. Finite – Time Ruin Probability in a Generalized Risk Processes under Interest Force with sequences of dependent random variables

Let model (1.1). We assume that:

Assumption 2.1. u, t are possitive integer numbers.

Assumption 2.2. $X = \left\{ X_n \right\}_{n\geq 1}$ is a homogeneous Markov chain, X_n take values in a finite set of positive numbers $G_X = \left\{ x_1, x_2, ..., x_M \right\} (0 < x_1 < x_2 < ... < x_M)$ with $p_{ij} = P\left(X_{m+1} = x_j \middle| X_m = x_i \right), (m \in N, x_i \in G_X, x_j \in G_X)$ where $0 \leq p_{ij} \leq 1, \sum_{j=1}^{M} p_{ij} = 1$.

In addition, $P(X_1 = x_i) = p_i \, (x_i \in G_X)$, $0 \leq p_i \leq 1, \sum_{i=1}^{M} p_i = 1$.

Assumption 2.3. $Y = \left\{ Y_n \right\}_{n\geq 1}$ is a homogeneous Markov chain, Y_n take values in a finite set of positive numbers $G_Y = \left\{ y_1, y_2, ..., y_N \right\} (0 < y_1 < y_2 < ... < y_N)$ with $q_{rs} = P\left(Y_{m+1} = y_s \middle| Y_m = y_r \right), (m \in N, y_r \in G_Y, y_s \in G_Y)$ where $0 \leq q_{rs} \leq 1, \sum_{s=1}^{N} q_{rs} = 1$.

In addition, $P(Y_1 = y_i) = q_i \, (y_i \in G_Y)$, $0 \leq q_i \leq 1, \sum_{i=1}^{N} q_i = 1$.

Assumption 2.4. $I = \left\{ I_n \right\}_{n\geq 1}$ is a homogeneous Markov chain, I_n take values in a finite set of non - negative numbers $G_I = \left\{ i_1, i_2, ..., i_R \right\} (0 \leq i_1 < i_2 < ... < i_R)$ with $r_{uv} = P\left(I_{m+1} = r_v \middle| Y_m = r_u \right), (m \in N, r_u \in G_Y, r_v \in G_Y)$ where $0 \leq r_{uv} \leq 1, \sum_{s=1}^{R} r_{uv} = 1$.

In addition, $r_k = P(I_1 = i_k)(i_k \in G_I)$, $0 \leq r_k \leq 1, \sum_{k=1}^{R} r_k = 1$.

Asumption 2.5. The sequences $\left\{X_n\right\}_{n\geq 1}$, $\left\{Y_n\right\}_{n\geq 1}$ and $\left\{I_n\right\}_{n\geq 1}$ are assumed to be independent.

From (1.1), we have:

$$U_t = u.\prod_{k=1}^{t}(1+I_k)+\sum_{k=1}^{t-1}\left((X_k - Y_k)\prod_{j=k+1}^{t}(1+I_j)\right)+X_t - Y_t. \qquad (2.1)$$

where throughout this paper, we denote $\prod_{t=a}^{b} z_t = 1$ and $\sum_{t=a}^{b} z_t = 0$ if $a > b$ and $A \overset{as}{=} B$ if $P(A\Delta B) = 0$ with $A\Delta B = \left(A \setminus B\right) \cup \left(B \setminus A\right)$.

Supposing that the ruin time is defined by $T_u = \inf\left\{j : U_j < 0\right\}$, where $\inf \phi = \infty$.

We define the finite time ruin (non-ruin) probabilities of model (1.1) with assumption 2.1 to assumption 2.4, respectively, by

$$\psi_t^{(1)}(u) = P(T_u \leq t) = P\left(\bigcup_{j=1}^{t}(U_j < 0)\right), \qquad (2.2)$$

$$\varphi_t^{(1)}(u) = 1 - \psi_t^{(1)}(u) = P(T_u \geq t+1) = P\left(\bigcap_{j=1}^{t}(U_j \geq 0)\right). \qquad (2.3)$$

Now, we give an exact formula for finite time ruin (non-ruin) probability of model (1.1).

Theorem 2.1. If model (1.1) satisfies assumptions 2.1 to 2.5, then finite time non-ruin probability of model (1.1) is defined by

$$\varphi_t^{(1)}(u) =$$
$$\sum_{c_1,c_2,\ldots,c_t=1}^{R} \sum_{m_1,m_2,\ldots,m_t=1}^{M} r_{c_1} r_{c_1c_2} \ldots r_{c_{t-1}c_t} p_{m_1} p_{m_1m_2} \ldots p_{m_{t-1}m_t} \left(\sum_{1\le n_1 \le g_1} \sum_{1\le n_2 \le g_2} \ldots \sum_{1\le n_t \le g_t} q_{n_1} q_{n_1n_2} \ldots q_{n_{t-1}n_t} \right), \quad (2.4)$$

where

$$g_1 = \max\left\{ n_1 : y_{n_1} \le \min\left\{ u\prod_{k=1}^{1}(1+i_{c_k}) + x_{m_1}, y_N \right\}\right\},$$

$$g_2 = \max\left\{ n_2 : y_{n_2} \le \min\left\{ u\prod_{k=1}^{2}(1+i_{c_k}) + \sum_{k=1}^{1}(x_{m_k} - y_{n_k}) \prod_{j=k+1}^{2}(1+i_{c_j}) + x_{m_2}, y_N \right\}\right\},$$

...

$$g_t = \max\left\{ n_t : y_{n_t} \le \min\left\{ u\prod_{k=1}^{t}(1+i_{c_k}) + \sum_{k=1}^{t-1}(x_{m_k} - y_{n_k}) \prod_{j=k+1}^{t}(1+i_{c_j}) + x_{m_t}, y_N \right\}\right\}$$

.

Proof.

Firstly, we have

$$A := \bigcap_{j=1}^{t} (U_j \ge 0)$$

$$= \left(Y_1 \le u\prod_{k=1}^{1}(1+I_k) + X_1 \right) \cap$$

$$\left(Y_2 \le u\prod_{k=1}^{2}(1+I_k) + \sum_{k=1}^{1}(X_k - Y_k)\prod_{j=k+1}^{2}(1+I_j) + X_2 \right) \cap$$

$$\left(Y_3 \le u\prod_{k=1}^{3}(1+I_k) + \sum_{k=1}^{2}(X_k - Y_k)\prod_{j=k+1}^{3}(1+I_j) + X_3 \right) \cap \ldots$$

$$\ldots \cap \left(Y_t \le u\prod_{k=1}^{t}(1+I_k) + \sum_{k=1}^{t-1}(X_k - Y_k)\prod_{j=k+1}^{t}(1+I_j) + X_t \right). \quad (2.5)$$

By assumption 2.4, we put $I_1 = i_{c_1}, I_2 = i_{c_2}, ..., I_t = i_{c_t}$ with $i_{c_1}, i_{c_2}, ..., i_{c_t}$ being non - negative numbers and statisfy condition: $0 \le i_{c_1}, i_{c_2}, ..., i_{c_t} \le i_R$.

Let $A_{c_1c_2...c_t} = \left(I_1 = i_{c_1}\right) \cap \left(I_2 = i_{c_2}\right) \cap ... \cap \left(I_t = i_{c_t}\right)$.

Since $I = \left\{I_n\right\}_{n\ge 1}$ is a homogeneous Markov chain then

$$P(A_{c_1c_2...c_t}) = P\left[\left(I_1 = i_{c_1}\right) \cap \left(I_2 = i_{c_2}\right) \cap ... \cap \left(I_t = i_{c_t}\right)\right]$$
$$= P\left(I_1 = i_{c_1}\right).P\left(I_2 = i_{c_2} \middle| I_1 = i_{c_1}\right)...P\left(I_t = i_{c_t} \middle| I_1 = i_{c_{t-1}}\right) = r_{c_1} r_{c_1c_2} ... r_{c_{t-1}c_t} \quad (2.6)$$

By Assumption 2.2, we put $X_1 = x_{m_1}, X_2 = x_{m_2}, ..., X_t = x_{m_t}$ with $x_{m_1}, x_{m_2}, ..., x_{m_t}$ being positive numbers and satisfy condition: $0 < x_{m_1}, x_{m_2}, ..., x_{m_t} \le x_M$.

Let $A_{m_1m_2...m_t} = \left(X_1 = x_{m_1}\right) \cap \left(X_2 = x_{m_2}\right) \cap ... \cap \left(X_t = x_{m_t}\right)$.

Since $X = \left\{X_n\right\}_{n\ge 1}$ is a homogeneous Markov chain then

$$P(A_{m_1m_2...m_t}) = P\left[\left(X_1 = x_{m_1}\right) \cap \left(X_2 = x_{m_2}\right) \cap ... \cap \left(X_t = x_{m_t}\right)\right]$$
$$= P\left(X_1 = x_{m_1}\right).P\left(X_2 = x_{m_2} \middle| X_1 = x_{m_1}\right)...P\left(X_t = x_{m_t} \middle| X_1 = x_{m_{t-1}}\right)$$
$$= p_{m_1} p_{m_1m_2} ... p_{m_{t-1}m_t} .$$
(2.7)

Firsly, we consider $I_1 = i_{c_1}\,(c_1 = \overline{1,R})$ then (2.5) is given

$$A \overset{as}{=} \bigcup_{c_1=1}^{R}\left(I_1 = i_{c_1}\right) \cap \left(\left(Y_1 \le u\prod_{c_1=1}^{1}(1+i_{c_1}) + X_1\right) \cap\right.$$

$$\left(Y_2 \le u(1+i_{c_1})\prod_{k=2}^{2}(1+I_k) + \sum_{k=1}^{1}(X_k - Y_k)\prod_{j=k+1}^{2}(1+I_j) + X_2\right) \cap$$

$$\left(Y_3 \le u(1+i_{c_1})\prod_{k=2}^{3}(1+I_k) + \sum_{k=1}^{2}(X_k - Y_k)\prod_{j=k+1}^{3}(1+I_j) + X_3\right) \cap \ldots$$

$$\left.\ldots \cap \left(Y_t \le u(1+i_{c_1})\prod_{k=2}^{t}(1+I_k) + \sum_{k=1}^{t-1}(X_k - Y_k)\prod_{j=k+1}^{t}(1+I_j) + X_t\right)\right)$$

Similarly, we consider $I_2 = i_{c_2}, \ldots, I_t = i_{c_t}\ (c_2, \ldots, c_t = \overline{1,R})$, (2.5) can be written as

$$A \overset{as}{=} \bigcup_{c_1, c_2, \ldots, c_t = 1}^{R} \left\{\left(I_1 = i_{c_1}\right) \cap \left(I_2 = i_{c_2}\right) \cap \ldots \cap \left(I_t = i_{c_t}\right)\right\} \cap$$

$$\left(\left(Y_1 \le u\prod_{k=1}^{1}(1+i_{c_k}) + X_1\right) \cap\right.$$

$$\left(Y_2 \le u\prod_{k=1}^{2}(1+i_{c_k}) + \sum_{k=1}^{1}(X_k - Y_k)\prod_{j=k+1}^{2}(1+i_{c_j}) + X_2\right) \cap$$

$$\left(Y_3 \le u\prod_{k=1}^{3}(1+i_{c_k}) + \sum_{k=1}^{2}(X_k - Y_k)\prod_{j=k+1}^{3}(1+i_{c_j}) + X_3\right) \cap \ldots$$

$$\left.\ldots \cap \left(Y_t \le u\prod_{k=1}^{t}(1+i_{c_k}) + \sum_{k=1}^{t-1}(X_k - Y_k)\prod_{j=k+1}^{t}(1+i_{c_j}) + X_t\right)\right).$$

Next, we consider $X_1 = x_{m_1}\,(m_1 = \overline{1,M})$, then

$$A \overset{as}{=} \bigcup_{c_1, c_2, \ldots, c_t = 1}^{R} \left(\left\{\left(I_1 = i_{c_1}\right) \cap \left(I_2 = i_{c_2}\right) \cap \ldots \cap \left(I_t = i_{c_t}\right)\right\}\right.$$

$$\cap\left(\bigcup_{m_1=1}^{M}(X_1 = x_{m_1}) \cap \left(\left(Y_1 \le u\prod_{k=1}^{1}(1+i_{c_k}) + x_{m_1}\right) \cap\right.\right.$$

$$\left(Y_2 \le u\prod_{k=1}^{2}(1+i_{c_k}) + \sum_{k=1}^{1}(x_{m_k} - Y_k)\prod_{j=k+1}^{2}(1+i_{c_j}) + X_2\right)\cap$$
$$\left(Y_3 \le u\prod_{k=1}^{3}(1+i_{c_k}) + \left[(x_{m_1} - Y_1) + \sum_{k=2}^{2}(X_k - Y_k)\right]\prod_{j=k+1}^{3}(1+i_{c_j}) + X_3\right)\cap\ldots$$
$$\ldots\cap\left(Y_t \le u\prod_{k=1}^{t}(1+i_{c_k}) + \left[(x_{m_1} - Y_1) + \sum_{k=2}^{t-1}(X_k - Y_k)\right]\prod_{j=k+1}^{t}(1+i_{c_j}) + X_t\right)\Bigg)\Bigg).$$

Similarly, we consider $X_2 = x_{m_2}, \ldots, X_t = x_{m_t}\ (m_2, \ldots, m_t = \overline{1, M})$, (2.5) can be rearranged as

$$A \overset{as}{=} \bigcup_{c_1, c_2, \ldots, c_t = 1}^{R} \Bigg(\Big\{\big(I_1 = i_{c_1}\big)\cap\big(I_2 = i_{c_2}\big)\cap\ldots\cap\big(I_t = i_{c_t}\big)\Big\}\cap$$
$$\left(\bigcup_{m_1, m_2, \ldots, m_t = 1}^{M}\Big\{(X_1 = x_{m_1})\cap(X_2 = x_{m_2})\cap\ldots\cap(X_t = x_{m_t})\Big\}\cap\right.$$
$$\left(\left(Y_1 \le u\prod_{k=1}^{1}(1+i_{c_k}) + x_{m_1}\right)\cap\right.$$
$$\left(Y_2 \le u\prod_{k=1}^{2}(1+i_{c_k}) + \sum_{k=1}^{1}(x_{m_k} - Y_k)\prod_{j=k+1}^{2}(1+i_{c_j}) + x_{m_2}\right)\cap$$
$$\left(Y_3 \le u\prod_{k=1}^{3}(1+i_{c_k}) + \sum_{k=1}^{2}(x_{m_k} - Y_k)\prod_{j=k+1}^{3}(1+i_{c_j}) + x_{m_3}\right)\cap\ldots$$
$$\ldots\cap\left(Y_t \le u\prod_{k=1}^{t}(1+i_{c_k}) + \sum_{k=1}^{t-1}(x_{m_k} - Y_k)\prod_{j=k+1}^{t}(1+i_{c_j}) + x_{m_t}\right)\Bigg)\Bigg)\Bigg)$$
$$\overset{as}{=} \bigcup_{c_1, c_2, \ldots, c_t = 1}^{R}\Bigg(\Big\{(I_1 = i_{c_1})\cap(I_2 = i_{c_2})\cap\ldots\cap(I_t = i_{c_t})\Big\}\cap$$
$$\bigcup_{m_1, m_2, \ldots, m_t = 1}^{M}\left(\Big\{(X_1 = x_{m_1})\cap(X_2 = x_{m_2})\cap\ldots\cap(X_t = x_{m_t})\Big\}\cap C_{c_1 c_2 \ldots c_t}^{m_1 m_2 \ldots m_t}\right)\Bigg)$$
$$\overset{as}{=} \bigcup_{c_1, c_2, \ldots, c_t = 1}^{R}\left(\bigcup_{m_1, m_2, \ldots, m_t = 1}^{M}\Big\{A_{i_{c_1} i_{c_2} \ldots i_{c_t}} \cap B_{x_{m_1} x_{m_2} \ldots x_{m_t}} \cap C_{c_1 c_2 \ldots c_t}^{m_1 m_2 \ldots m_t}\Big\}\right), \qquad (2.8)$$

Where

$$C_{c_1c_2\ldots c_t}^{m_1m_2\ldots m_t} \overset{as}{=} \left(Y_1 \le u\prod_{k=1}^{1}(1+i_{c_k}) + x_{m_1} \right)$$

$$\cap \left(Y_2 \le u\prod_{k=1}^{2}(1+i_{c_k}) + \sum_{k=1}^{1}(x_{m_k} - Y_k)\prod_{j=k+1}^{2}(1+i_{c_j}) + x_{m_2} \right) \cap$$

$$\left(Y_3 \le u\prod_{k=1}^{3}(1+i_{c_k}) + \sum_{k=1}^{2}(x_{m_k} - Y_k)\prod_{j=k+1}^{3}(1+i_{c_j}) + x_{m_3} \right) \cap \ldots$$

$$\ldots \cap \left(Y_t \le u\prod_{k=1}^{t}(1+i_{c_k}) + \sum_{k=1}^{t-1}(x_{m_k} - Y_k)\prod_{j=k+1}^{t}(1+i_{c_j}) + x_{m_t} \right). \quad (2.9)$$

By assumption 2.3, we put $Y_1 = y_{n_1}, Y_2 = y_{n_2}, \ldots, Y_{t-1} = y_{n_{t-1}}$ with $y_{n_1}, y_{n_2}, \ldots, y_{n_{t-1}}$ being positive numbers and satisfy condition: $0 < y_{n_1}, y_{n_2}, \ldots, y_{n_{t-1}} \le y_N$.

Thus, (2.9) can be written as

$$C_{m_1m_2\ldots m_t}^{c_1c_2\ldots c_t} \overset{as}{=} \bigcup_{y_{n_1} \le u\prod_{k=1}^{1}(1+i_{c_k})+x_{m_1}} \left(Y_1 = y_{n_1} \right) \cap$$

$$\left(\left(Y_2 \le u\prod_{k=1}^{2}(1+i_{c_k}) + \sum_{k=1}^{1}(x_{m_k} - y_{n_k})\prod_{j=k+1}^{2}(1+i_{c_j}) + x_{m_2} \right) \cap \right.$$

$$\left(Y_3 \le u\prod_{k=1}^{3}(1+i_{c_k}) + \left[(x_{m_1} - y_{n_1}) + \sum_{k=2}^{2}(x_{m_k} - y_{n_k}) \right]\prod_{j=k+1}^{3}(1+i_{c_j}) + x_{m_3} \right) \cap \ldots$$

$$\left. \ldots \cap \left(Y_t \le u\prod_{k=1}^{t}(1+i_{c_k}) + \left[(x_{m_1} - y_{n_1}) + \sum_{k=2}^{t-1}(x_{m_k} - y_{n_k}) \right]\prod_{j=k+1}^{t}(1+i_{c_j}) + x_{m_t} \right) \right)$$

$$\overset{as}{=} \bigcup_{y_{n_1} \le \left(u \prod_{k=1}^{1} (1+i_{c_k}) + x_{m_1} \right)} (Y_1 = y_{n_1}) \cap$$

$$\left(\bigcup_{y_{n_2} \le \left(u \prod_{k=1}^{2} (1+i_{c_k}) + \sum_{k=1}^{1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{2} (1+i_{x_j}) + x_{m_2} \right)} (Y_2 = y_{n_2}) \cap \right.$$

$$\left(\bigcup_{y_{n_3} \le \left(u \prod_{k=1}^{3} (1+i_{c_k}) + \sum_{k=1}^{2} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{3} (1+i_{x_j}) + x_{m_3} \right)} (Y_3 = y_{n_3}) \cap \ldots \right.$$

$$\left. \left. \ldots \cap \left(Y_t \le u \prod_{k=1}^{t} (1+i_{c_k}) + \sum_{k=1}^{t-1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{t} (1+i_{c_j}) + x_{m_t} \right) \ldots \right) \right). \qquad (2.10)$$

Using by assumption 2.3, we put $Y_t = y_{n_t}$ with y_{n_t} being positive number and statisfy condition $0 < y_{n_t} \le y_N$ then (2.9) can be rearranged as

$$C_{m_1 m_2 \ldots m_t}^{c_1 c_2 \ldots c_t} \overset{as}{=} \bigcup_{y_{n_1} \le \left(u \prod_{k=1}^{1} (1+i_{c_k}) + x_{m_1} \right)} \left(\bigcup_{y_{n_2} \le \left(u \prod_{k=1}^{2} (1+i_{c_k}) + \sum_{k=1}^{1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{2} (1+i_{c_j}) + x_{m_2} \right)} \left(\bigcup_{y_{n_3} \le \left(u \prod_{k=1}^{3} (1+i_{c_k}) + \sum_{k=1}^{2} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{3} (1+i_{c_j}) + x_{m_3} \right)} \ldots \right. \right.$$

$$\left. \left. \ldots \left(\bigcup_{y_{n_t} \le \left(u \prod_{k=1}^{t} (1+i_{c_k}) + \sum_{k=1}^{t-1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{t} (1+i_{c_j}) + x_{m_t} \right)} \left\{ (Y_1 = y_{n_1}) \cap \ldots \cap (Y_t = y_{n_t}) \right\} \ldots \right) \right) \right) \qquad (2.11)$$

By using Lemma 2.1 in [6], $u \prod_{k=1}^{1} (1+i_{c_k}) + x_{m_1}$,

$$u\prod_{k=1}^{2}(1+i_{c_k})+\sum_{k=1}^{1}(x_{m_k}-y_{n_k})\prod_{j=k+1}^{2}(1+i_{c_j})+x_{m_2}\ , \qquad \ldots,$$

$u\prod_{k=1}^{t}(1+i_{c_k})+\sum_{k=1}^{t}(x_{m_k}-y_{n_k})\prod_{j=k+1}^{t}(1+i_{c_j})+x_{m_t}$ are positive numbers and $0<y_{n_1},y_{n_2},\ldots,y_{n_t}\le y_N$ then, we define

$$g_1=\max\left\{n_1 : y_{n_1}\le\min\left\{u\prod_{k=1}^{1}(1+i_{c_k})+x_{m_1},y_N\right\}\right\},$$

$$g_2=\max\left\{n_2 : y_{n_2}\le\min\left\{u\prod_{k=1}^{2}(1+i_{c_k})+\sum_{k=1}^{1}(x_{m_k}-y_{n_k})\prod_{j=k+1}^{2}(1+i_{c_j})+x_{m_2},y_N\right\}\right\},$$

...

$$g_t=\max\left\{n_t : y_{n_t}\le\min\left\{u\prod_{k=1}^{t}(1+i_{c_k})+\sum_{k=1}^{t-1}(x_{m_k}-y_{n_k})\prod_{j=k+1}^{t}(1+i_{c_j})+x_{m_t},y_N\right\}\right\}$$

.

Thus, (2.11) can be rearranged as

$$C^{c_1c_2\ldots c_t}_{m_1m_2\ldots m_t}\overset{as}{=}\bigcup_{1\le n_1\le g_1}\bigcup_{1\le n_2\le g_2}\cdots\bigcup_{1\le n_t\le g_t}\left\{\left(Y_1=y_{n_1}\right)\cap\left(Y_2=y_{n_2}\right)\cap\ldots\cap\left(Y_t=y_{n_t}\right)\right\}. \quad (2.12)$$

Because $Y=\left\{Y_n\right\}_{n\ge1}$ is a homogeneous Markov chain then

$$P\left[\left(Y_1=y_{n_1}\right)\cap\left(Y_2=y_{n_2}\right)\cap\ldots\cap\left(Y_t=y_{n_t}\right)\right]$$

$$=P\left(Y_1=y_{n_1}\right).P\left(Y_2=y_{n_2}\middle|Y_1=y_{n_1}\right)\ldots P\left(Y_t=y_{n_t}\middle|Y_1=y_{n_{t-1}}\right)=q_{n_1}q_{n_1n_2}\ldots q_{n_{t-1}n_t}$$

In the other hand, system of events $\left\{\left(Y_1=y_{n_1}\right)\cap\left(Y_2=y_{n_2}\right)\cap\ldots\cap\left(Y_t=y_{n_t}\right)\right\}_{1\le n_j\le g_j\ (j=\overline{1,t})}$ in (2.12) be incompatible then

$$P(B_{m_1m_2\ldots m_t})=\sum_{1\le n_1\le g_1}\sum_{1\le n_2\le g_2}\cdots\sum_{1\le n_t\le g_t}q_{n_1}q_{n_1n_2}\ldots q_{n_{t-1}n_t}\ . \quad (2.13)$$

By X, Y, I **are assumed to be independent, with** $c_1, c_2, ..., c_t$ **and** $m_1, m_2, ..., m_t$ **hold then**

$A_{c_1c_2...c_t}, B_{m_1m_2...m_t}, C_{c_1c_2...c_t}^{m_1m_2...m_t}$ **are independent events.**

In addition, system of events $\left\{A_{c_1c_2...c_t} \cap B_{m_1m_2...m_t} \cap C_{c_1c_2...c_t}^{m_1m_2...m_t}\right\}_{c_j=\overline{1,R};\, m_j=\overline{1,M}\ (\overline{j=1,t})}$ **in (2.8) is incompatible.**

Therefore, using (2.8) combining (2.6), (2.7) and (2.13), we have

$$
\begin{aligned}
\varphi_t^{(1)}(u) &= P(A) \\
&= \sum_{c_1,c_2,..,c_t=1}^{R} \left(\sum_{m_1,m_2,...,m_t=1}^{M} P\left\{A_{c_1c_2...c_t} \cap B_{m_1m_2...m_t} \cap C_{c_1c_2...c_t}^{m_1m_2...m_t}\right\} \right) \\
&= \sum_{c_1,c_2,...,c_t=1}^{R} \left(\sum_{m_1,m_2,...,m_t=1}^{M} P\left(A_{c_1c_2...c_t}\right).P\left(B_{m_1m_2...m_t}\right).P\left(C_{c_1c_2...c_t}^{m_1m_2...m_t}\right) \right) \\
&= \sum_{c_1,c_2,..,c_t=1}^{R} \sum_{m_1,m_2,...,m_t=1}^{M} P\left(A_{c_1c_2...c_t}\right).P\left(B_{m_1m_2...m_t}\right).
\end{aligned}
$$

$$
\left(\sum_{1\le n_1\le g_1} \sum_{1\le n_2\le g_2} .... \sum_{1\le n_t\le g_t} q_{n_1} q_{n_1n_2} ... q_{n_{t-1}n_t} \right)
$$

$$
= \sum_{c_1,c_2,..,c_t=1}^{R} \sum_{m_1,m_2,...,m_t=1}^{M} r_{c_1} r_{c_1c_2} ... r_{c_{t-1}c_t} p_{m_1} p_{m_1m_2} ... p_{m_{t-1}m_t} \ .
$$

$$
\left(\sum_{1\le n_1\le g_1} \sum_{1\le n_2\le g_2} ... \sum_{1\le n_t\le g_t} q_{n_1} q_{n_1n_2} ... q_{n_{t-1}n_t} \right). \tag{2.14}
$$

This completes the proof of the Theorem 2.1.

Corollary 2.1. If model (1.1) satisfies assumptions 2.1 to 2.4, then finite time ruin probability of model (1.1) is defined by

$$\psi_t^{(1)}(u) = 1 - \varphi_t^{(1)}(u)$$
$$= \sum_{c_1,c_2,\dots,c_t=1}^{R} \sum_{m_1,m_2,\dots,m_t=1}^{M} r_{c_1} r_{c_1c_2} \dots r_{c_{t-1}c_t} p_{m_1} p_{m_1m_2} \dots p_{m_{t-1}m_t} \left(\sum_{1\le n_1\le g_1} \sum_{1\le n_2\le g_2} \dots \sum_{1\le n_t\le g_t} q_{n_1} q_{n_1n_2} \dots q_{n_{t-1}n_t} \right). \quad (2.15)$$

Remark 2.1.

Fomula (2.4) (or (2.15) gives a method to compute axactly finite time non-ruin (ruin) probability of model (1.1) which $X = \{X_n\}_{n\ge1}$ and $Y = \{Y_n\}_{n\ge1}$ are homogeneous Markov chains, they take values in a finite set of positive numbers and $I = \{I_n\}_{n\ge1}$ is a homogeneous Markov chain and they take values in a finite set of non- negative numbers.

Let model (1.2) satisfy assumptions 2.1 to 2.5.

From (1.2), we have:

$$U_t = u.\prod_{k=1}^{t}(1+I_k) + \sum_{k=1}^{t-1}\left((X_k(1+I_k) - Y_k)\prod_{j=k+1}^{t}(1+I_j) \right) + X_t - Y_t \quad (2.16)$$

Supposing that the ruin time of model (1.2) is defined by $T_u = \inf\{ j : U_j < 0\}$, where $\inf \phi = \infty$.

We define the finite time ruin (non-ruin) probabilities of model (1.2) with ssumptions 2.1 to 2.5, respectively, by

$$\psi_t^{(2)}(u) = P(T_u \le t) = P\left(\bigcup_{k=1}^{t}(U_k < 0) \right), \quad (2.17)$$

$$\varphi_t^{(2)}(u) = 1 - \psi_t^{(2)}(u) = P(T_u \ge t+1) = P\left(\bigcap_{k=1}^{t}(U_k \ge 0) \right). \quad (2.19)$$

Next, we give an exact formula for finite time ruin (non ruin) probability of model (1.2).

Theorem 2.2. If model (1.2) satisfies assumptions 2.1 to 2.5, then finite time non-ruin probability of model (1.2) is defined by

$$\varphi_t^{(2)}(u)=\sum_{c_1,c_2,..,c_t=1}^{R}\ \sum_{m_1,m_2,...,m_t=1}^{M} r_{c_1} r_{c_1c_2}...r_{c_{t-1}c_t}\, p_{m_1} p_{m_1m_2}...p_{m_{t-1}m_t}\,.$$

$$\left(\sum_{1\le n_1\le g_1}\ \sum_{1\le n_2\le g_2}...\sum_{1\le n_t\le g_t} q_{n_1} q_{n_1n_2}...q_{n_{t-1}n_t}\right), \qquad (2.20)$$

where

$$g_1=max\left\{n_1: y_{n_1}\le \min\left\{u\prod_{k=1}^{1}(1+i_{c_k})+x_{m_1}(1+i_{c_1}),y_N\right\}\right\},$$

$$g_2=max\left\{n_2: y_{n_2}\le \min\left\{u\prod_{k=1}^{2}(1+i_{c_k})+\sum_{k=1}^{1}(x_{m_k}(1+i_{c_k})-y_{n_k})\prod_{j=k+1}^{2}(1+i_{c_j})+x_{m_2}(1+i_{c_2}),y_N\right\}\right\},$$

$$g_t=max\left\{n_t: y_{n_t}\le \min\left\{u\prod_{k=1}^{t}(1+i_{c_k})+\sum_{k=1}^{t-1}(x_{m_k}(1+i_{c_k})-y_{n_k})\prod_{j=k+1}^{t}(1+i_{c_j})+x_{m_t}(1+i_{c_t}),y_N\right\}\right\}.$$

Proof.

We proof similarly as Theorem 2.1.

Corollary 2.2. If model (1.2) satisfies assumptions 2.1 to 2.5, then finite time ruin probability of model (1.2) is defined by

$$\psi_t^{(2)}(u)=1-\varphi_t^{(2)}(u)$$

$$=1-\sum_{c_1,c_2,..,c_t=1}^{R}\ \sum_{m_1,m_2,...,m_t=1}^{M}(r_{c_1}r_{c_1c_2}...r_{c_{t-1}c_t}).(p_{m_1}p_{m_1m_2}...p_{m_{t-1}m_t})\left(\sum_{1\le n_1\le g_1}\ \sum_{1\le n_2\le g_2}...\sum_{1\le n_t\le g_t} q_{n_1}q_{n_1n_2}...q_{n_{t-1}n_t}\right) \qquad (2.21)$$

.

Remark 2.2. Fomula (2.20) (or (2.21)) give a method to compute exact finite time non-ruin (ruin) probability of model (1.2) which $X=\{X_n\}_{n\geq 1}$ and $Y=\{Y_n\}_{n\geq 1}$ are homogeneous Markov chains and they take values in a finite set of positive numbers. In addition, $I=\{I_n\}_{n\geq 1}$ is a homogeneous Markov chain and they take values in a finite set of non- negative numbers.

3. Conclusion

Using technique of classical probability with u, t are possitive integer numbers, claims and premiums which all are positive numbers and interests are non – negative numbers, this paper constructed an exact formula for ruin (non-ruin) probability for model (1.1) and model (1.2) where sequences of claims, premiums and interests are homogeneous Markov chains. Our main results in this paper are Theorem 2.1 and Theorem 2.2.

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