Generating Functions for $X(n)$ and $Y(n)$

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Abstract

This paper shows how to prove the Theorem $X(n)=Y(n)$, i.e., the number of partitions of n with no part repeated more than twice is equal to the number of partitions of n with no part is divisible by 3.

Key Words: Infinite factors, enumerated by $X(n)$

1. Introduction

We give some definitions of $X(n)$ and $Y(n)$ [1]. We generate the generating functions for $X(n)$ and $Y(n)$, and prove the Theorem $X(n)=Y(n)$. Finally we give a numerical example when $n = 8$.

2. Definitions

$X(n)$: The number of partitions of n with no part repeated more than twice.
$Y(n)$: The number of partitions of n with no part is divisible by 3.

3. Generating Functions

We consider a function, which is the product of infinite factors, one of which is $\left(1+x^n+x^{2n}\right)$ and it can be written as;

$$=\left(1+x+x^2\right)\left(1+x^2+x^4\right)\left(1+x^3+x^6\right)\ldots\infty$$
$$=1+x+2x^2+2x^3+4x^4+5x^5+7x^6+\ldots\infty$$
$$=1+\sum_{n=1}^{\infty}X(n)x^n \tag{1}$$

Each element of the product comes from multiplying together one term from each bracket either x^0 or x^n or x^{n+n} from $\left(1+x^n+x^{2n}\right)$. So in the corresponding partitions no part occurs more than twice.

Therefore we can say that the coefficient $X(n)$ of x^n in the above expansion is the number of partitions of n with no part is repeated more than twice.

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The generating function for $Y(n)$ is of the form [2];

$$\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)\ldots(1-x^{3n-2})(1-x^{3n-1})\ldots}$$
$$=1+x+2x^2+2x^3+4x^4+5x^5+7x^6+\ldots\infty$$
$$=1+\sum_{n=1}^{\infty}Y(n)x^n \qquad (2)$$

where the coefficient $Y(n)$ is the number of partitions of *n* with no part is divisible by 3.

From equations (1) and (2) we get;

$$=1+\sum_{n=1}^{\infty}X(n)x^n$$
$$=(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\ldots(1+x^n+x^{2n})\ldots\infty$$
$$=\frac{1-x^3}{1-x}.\frac{1-x^6}{1-x^2}.\frac{1-x^9}{1-x^3}\cdots\frac{1-x^{3n}}{1-x^n}\cdots$$
$$=\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)\ldots(1-x^{3n-2})(1-x^{3n-1})\ldots}$$
$$=1+\sum_{n=1}^{\infty}Y(n)x^n.$$

Equating the coefficient of x^n from both sides we get;

$X(n)=Y(n)$.

Theorem: $X(n)=Y(n)$, i.e., the number of partitions of *n* with no part is repeated more than twice is equal to the number of partitions of *n* with no part is divisible by 3.

Proof: We develop a one-to-one correspondence between the partitions enumerated by $X(n)$ and those enumerated by $Y(n)$. Let $n=a_1+a_2+\ldots+a_r$ be a partition of *n* with no part is repeated more than twice. We transfer this into a partition of *n* with no part is divisible by 3. If a part a_m of the partition, which is divisible by 3, enumerated by $X(n)$ can be expressed into three equal parts, such that: 6 = 2+2+2, 3 = 1+1+1. Rearranging the parts of the partition, we can say that the parts are not divisible by 3. Clearly, our correspondence is one-to-one.

Conversely, we start any partition of *n* into with no part is divisible by 3, say $n=a_1+a_2+\ldots+a_r$, we consider the same part not less than thrice, it would be unique sum by same three parts by taking a group, such that, 5+1+1+1 = 5+3 and 2+2+2+1+1 = 6+1+1.

This gives *n* as a partition with no part is repeated more than twice. Thus, we have the one-to-one correspondence. The corresponding is onto, so that $X(n)=Y(n)$. Hence the Theorem.

4. A Numerical Example

When $n = 8$, the listed partitions of 8 with no part repeated more than twice is given below:

8 = 7+1 = 6+2 = 6+1+1 = 5+3 = 5+2+1 = 4+4 = 5+3+1 = 4+2+1+1 = 4+2+2 = 3+3+2 = 3+3+1+1= 3+2+2+1.

So, there are 13 partitions i.e., $X(8) = 13$. Again, the list of partitions of 8 with no part is divisible by 3 is given below:

8 = 7+1 = 5+2+1 = 5+1+1+1+1 = 4+4 = 4+2+1+1 = 4+2+2= 4+1+1+1+1 = 2+2+2+2 =2+2+2+1+1= 2+2+1+1+1+1 = 2+1+1+1+1+1+1 = 1+1+1+1+1+1+1+1.

So, there are 13 partitions i.e., $Y(8) = 13$.

$\therefore X(n) = Y(n)$

5. Conclusion

For any positive integer of n, we can verify the Theorem $X(n) = Y(n)$. We have already satisfied the Theorem when $n = 8$.

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