An Integral Related to the Weibull, Inverse Weibull and Bur XII Distributions

M. Aleem[1]

Abstract

This paper gives some integrals which are worthful for the constitution of Fisher information for Bur XII, Pareto, Weibull, Inverse Weibull and related Distributions. Some work has been done by Baazauskas (2003) and Alan (2011), we have given more generalizations.

Key Words: Bur XII, Weibull, Inverse Weibull distributions, Expectation, Integration, Differentiation

1. Preliminaries

Let we consider the integral given as:

$$\int_0^\infty \frac{(\log x)^m}{\left(1+\frac{x^c}{\lambda}\right)^b} dx \qquad (1.1)$$

Where "b, c and $\lambda > 0$" and "m" is non-negative integer, for a = 1 and $\lambda = 1$ (1.1) become same as Alan (2011). As noted by Baazauskar (2003) that such type of integrals used in Fisher Information for Beta, Pareto, Bur and related distributions.

The objective of this paper is to present the generalization of the Alan (2011) results.

a. The cumulative distribution function (cdf) of three parameter Burr XII distribution is given as:

Where $x \geq 0$, shape parameter b, c > 0 and scale parameter $\lambda > 0$

$$F(x;b,c,\lambda)=1-\left(1+\frac{x^c}{\lambda}\right)^{-b} \qquad (1.2)$$

---

[1] Former Professor , Department of Statistics, The Islamia University of Bahawalpur, Pakistan.

b. When c=1 in eq. (1.2) the resultant distribution is known as Lomax, see Lomax (1954) for an application to the analysis of business failure data, the cumulative distribution function (cdf) is given as:

$$F(x,b,\lambda)=1-\left(1+\frac{x}{\lambda}\right)^{-b} \qquad (1.3)$$

$$\int_0^\infty \frac{(\log x)^m}{\left(1+\frac{x^c}{\lambda}\right)^b}\,dx = \frac{\lambda^{\frac{1}{c}}}{c}\int_0^\infty \frac{\left\{\log\left(x^{\frac{1}{c}}y^{\frac{1}{c}}\right)\right\}^m y^{\frac{1}{c}-1}}{(1+y)^b}\,dy$$

$$= \frac{\Gamma(\frac{1}{c})\Gamma(b-\frac{1}{c})}{\Gamma b}\ E\left(\log\ \lambda^{\frac{1}{c}}y^{\frac{1}{c}}\right)^m \qquad (1.4)$$

Now by putting $\dfrac{x^c}{\lambda} = y$ in eq. (1.1), we get

Here "E" denotes the expectation w.r.t the Bur-xii   distribution given in eq. (1.2) that we can write (1.4) as:

$$=\frac{\Gamma(\frac{1}{c})\Gamma(b-\frac{1}{c})}{\Gamma b}\frac{d^m}{d_r^m}\left\{\lambda^{\frac{r}{c}}E\left(y^{\frac{1}{c}}\right)^r\right\}_{/r=0} \qquad (1.5)$$

In eq. (1.5) each of the "m" differentiations w.r.t generates a factor "log x", and moments of Bur distribution given in (1.2) are as follows:

$$E\left(y^{\frac{1}{c}}\right)^r = \beta\left(\frac{r}{c}+\frac{1}{c}, b-\frac{r}{c}-\frac{1}{c}\right)$$

$$= \frac{\Gamma\left(\frac{r}{c}+\frac{1}{c}\right),\Gamma\left(b-\frac{r}{c}-\frac{1}{c}\right)}{\Gamma b} \qquad (1.6)$$

Where "β (.)" and "Γ (.)" are beta and gamma functions respectively, see Watkins (1997).

2. Derivation of the Results

$$\int_0^\infty \frac{(\log x)^m}{\left(1+\dfrac{x^c}{\lambda}\right)^b}\,dx = \frac{\Gamma(\dfrac{1}{c})\Gamma(b-\dfrac{1}{c})}{(\Gamma b)^2}\frac{d^m}{d_r^m}$$

$$\left\{\lambda^{\frac{r}{c}}\Gamma\left(\frac{r}{c}+\frac{1}{c}\right),\Gamma\left(b-\frac{r}{c}-\frac{1}{c}\right)\right\}_{/r=0}$$

a. By putting the result of eq. (1.6) in eq. (1.5),we get as:

Now differentiating "m" times and then putting "r = 0"; writing $\Gamma^{(i)}$ for the ith derivative of Γ, we get as:

$$\int_0^\infty \frac{(\log x)^m}{\left(1+\dfrac{x^c}{\lambda}\right)^b}\,dx = \frac{\Gamma(\dfrac{1}{c})\;\Gamma(b-\dfrac{1}{c})}{(\Gamma b)^2}\frac{\lambda^{\frac{1}{c}}}{c^{m+1}}\sum_{i_1=0}^{m}\sum_{i_2=0}^{m-i_1}\binom{m}{i_1\;\;i_2}(-1)^{i_1}$$

$$(\log\lambda)^{i_1}\;\Gamma^{(i_1)}\left(b-\frac{1}{c}\right)\;\Gamma^{(m-i_1-i_2)}\left(\frac{1}{c}\right) \qquad\qquad (2.1)$$

$$i_1 = 0,1,2,\ldots m, \qquad i_2 = 0,1,2,\ldots(m-i_1)$$

For "c,λ = 1" and "b = d+1" The resultant (2.1) is same as in Brazaaskas (2003).

b. Now considering weibull distribution with cumulative distribution function (cdf), given as:

$$F(x) = 1-\exp\left(-\frac{x^\beta}{\gamma}\right), \qquad\qquad \beta,\gamma > 0,\;\; x \ge 0$$

And

From (1.5) along with (b) using eq. (2.2), we have

$$E(x)^r = \gamma^{\frac{r}{\beta}}\;\Gamma\left(1+\frac{r}{\beta}\right) \qquad\qquad\qquad (2.2)$$

$$E(\log x)^m = \left[\frac{d^m}{d_r^m}E(x)^r\right]_{r=0} \qquad\qquad\qquad (2.3)$$

By applying (2.2) in (2.3) we obtain:

$$\mathrm{E}(\log\ x)^m = \sum_{j=0}^{m} \binom{m}{j} \beta^{-m} \left(\log \gamma\right)^j \ \overset{(m-j)}{\Gamma}(1) \qquad\qquad (2.4)$$

$$j = 0,1,\ 2,\dots\ m$$

Now differentiating "m" times, then putting "r = 0" we obtain:

$$F(x) = \exp\left(\frac{1}{\gamma\,x^\beta}\right), \qquad\qquad \gamma,\ \beta > 0,\ \ x \geq 0 \qquad (2.5)$$

Similarly for Inverse weibull distribution with (cdf) given as:

$$\mathrm{E}(\log x)^m = \sum_{j=0}^{m} \binom{m}{j} \beta^{-m} \left(\log \gamma\right)^j \ \overset{(m-j)}{\Gamma}(1) \qquad\qquad (2.6)$$

$$j = 0,1,\ 2,\dots\ m$$

3. References

Brazauskas, V. (2003). "Information matrix for Pareto (IV), Bur, and related distributions" Commun.Statist.Theor.Meth.32:315-325.

Lomax, K, S. (1954). "Business failure: Another example of the analysis of failure data" J.Amer.Statist.Assoc.49:847-852.

Watkins,A.J. (1997). "A notse on expected Fisher information for the Burr XII distribution. Microelectronics and Reliability" 37:1849-1852.

Watkins, A .J. (1998). "On expectations associated with maximum likelihood estimation in the weibull distribution" J. Italian Statistical Society 7:15-26.

Alan J.Watkins (2011): "On an integral Related to the Burr XII distribution" Communications in Statistics-Theory and Methods,40:21, 3777-3779.

Mervat Mahdy Ramadan "A Class of Weighted Weibull Distributions and Its Properties"Studies in Mathematical Sciences Vol. 6, No. 1, 2013, pp. [35{45].