A Class of Modified Weighted Weibull Distribution and Its Properties

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1. Introduction

The weibull distribution and other related distribution like exponential, Rayleigh and extreme value distributions are very use full in survival, reliability renewal theory and branching processes can be seen in recent papers among the others Olnyede (2006) who shows that the weighted distribution are used to adjust the probabilities of the events as observed and recorded Gupta and Kundu (2009) discussed a new class of weighted weibull distributions.

2. Modified weighted weibull distribution.

Let x be a non-negative random variable with continuous distribution function F(x) and probability density function f(x) and X|{w}| be a weighted random variable with probability density function given as:

$$f_{X\{W\}}(x) = \frac{\left[1 - w(t(x))\right]^{c} f_X(x)}{\mathrm{E}\left[1 - w(t(x))\right]^{c}} \qquad -\infty < x < \infty$$

Where $0 < E\,[1\text{-}w\,(t(x))] < \infty$, Among others see; Ramadon (2013)

The new class of modified weighted distributions is defined by the probability density function (pdf) given as:

$$f_{X\,|\,\{\theta\}}(x) = \frac{\left(1 - F_X(\theta X)\right)^{c} f_X(x)}{\mathrm{E}\left(1 - F_X(\theta X)\right)^{c}}, \qquad x > 0 \qquad (2.1)$$

Where $F_x(\theta x) = w\,(t(x))$, and c, θ are additional shape parameters.

Now, the pdf of weibull distribution is given as:

$$f_{X\,|\,\{\beta,\gamma\}} = \beta\,\gamma\,x^{\gamma-1} e^{-\beta x^{\gamma}} \qquad (2\,.\,2)$$

Where "β" is scale parameter and "ϒ" is shape parameter.

The distribution function of X| {β, ϒ} is given as:

$$F_{X|\{\beta,\gamma\}}(x) = 1 - e^{-\beta x^{\gamma}} \qquad (2\,.\,3)$$

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And we get

$$F_{X|\{\beta,\gamma\}}(\theta x) = 1 - e^{-\beta(\theta x)^{\gamma}} \tag{2.4}$$

And

$$\mathrm{E}\left[1 - F_{X|\{\beta,\gamma\}}(\theta x)\right]^{c} = \int_{o}^{\infty}\left[1 - F_{X|\{\beta,\gamma\}}(\theta x)\right]^{C} f_{X|\{\beta,\gamma\}}(x)\,dx$$

$$= \frac{1}{c\theta^{\gamma}+1} \tag{2.5}$$

Putting (2.2) to (2.4) in (2.1) we get,

For x > 0

$$f_{X|\{\beta,\gamma,\theta,c\}}(x) = \beta\gamma(c\theta^{\gamma}+1)x^{\gamma-1}e^{-\beta(c\theta^{\gamma}+1)x^{\gamma}} \tag{2.6}$$

The pdf (2.6) is referred to as MWW (β, ϒ, θ, C).

Let X_1 and X_2 are two i.i.d distributed random variables with PDF f(y) and CDF F(y), then for θ > 0 and ϒ = 2 Mohdy (2011) worked on the weighted gamma distribution from two i.i.d gamma distribution. But we explore a new modified weighted gamma distribution from two i.i.d gamma distributions as given:

$$f_{x|\{\beta,2,\theta,c\}} = \frac{\left[1 - F_{y}(\theta x)\right]^{C} f_{y}(x)}{P_{x_1,x_2}\left(\left\{(x_1, x_2) : \sqrt{c}\,\theta x_1 < x_2\right\}\right)} \tag{2.7}$$

Now (2.1) can be obtained from eq. (2.7) by inserting

$$f_{X|\{\beta,2\}} = 2\beta x e^{-\beta x^2}$$

$$F_{X|\{\beta,2\}} = 1 - e^{-\beta x^2}$$

$$P(\sqrt{c}\,\theta x_1 > x_2) = \int_0^{\infty}\int_0^{\sqrt{c}\,\theta x_1} 4\beta^2 x_1 x_2 e^{-\beta(x_1^2+x_2^2)}dx_1 dx_2$$

$$= \frac{c\theta^2}{1+c\theta^2}$$

And

$$P(\sqrt{c}\,\theta x_1 < x_2) = 1 - P(\sqrt{c}\,\theta x_1 > x_2)$$

$$= \frac{1}{1+c\theta^2} \tag{2.8}$$

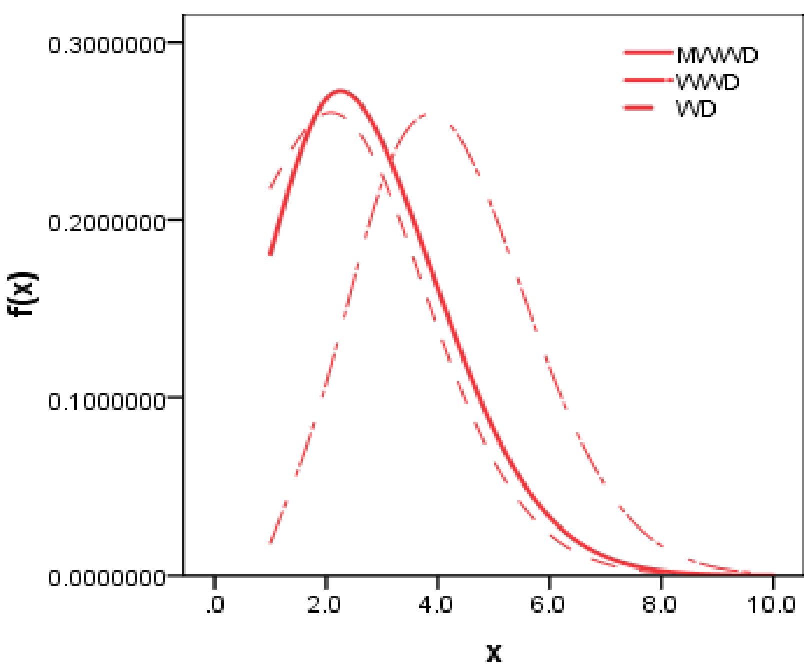
3. The Graph and Table

The table and graph of the MWW (β, ϒ, θ, C) by using eq. (2.6), WW (β, ϒ, θ) and W (β, ϒ) distribution functions are given below :

Table

f(x)	β=0.1, ϒ=2, θ=0.1, C=10		
X	MWWD	WWD	WD
1	0.180967	0.018269	0.217917
2	0.268128	0.108107	0.260191
3	0.243942	0.220748	0.225175
4	0.161517	0.258935	0.139012
5	0.082085	0.204695	0.064567
6	0.032788	0.117098	0.023104
7	0.010425	0.050351	0.006451
8	0.002658	0.016646	0.001416
9	0.000546	0.004294	0.000245
10	9.08E-05	0.000873	3.37E-05

Graph



Comments

We conclude that the new (pdf) MWW (β, ϒ, θ, C) is more skewed as compare to the WW (β, ϒ, θ) and W (β, ϒ) distribution functions.

4. Properties

i) The r^{th} moment of X|{β, ϒ, θ, C} is given as:

$$\mu'_r = \gamma(c\theta^{\gamma}+1)\beta^{-r/\gamma}\Gamma(r/\gamma+1)(c\theta^{\gamma}+1)^{-r/\gamma-1} \tag{4.1}$$

From (4.1) putting r = 1, r = 2 we get the mean and variance as:

$$Mean = \mu'_1 = \gamma(c\theta^{\gamma}+1)\beta^{-1/\gamma}\Gamma(1/\gamma+1)(c\theta^{\gamma}+1)^{-1/\gamma-1}$$

$$\mu'_2 = \gamma(c\theta^{\gamma}+1)\beta^{-2/\gamma}\Gamma(2/\gamma+1)(c\theta^{\gamma}+1)^{-2/\gamma-1}$$

And

$$\mathrm{var}(x) = \mu_2 - \mu_1'^2$$

$$S.\ D = \sqrt{Variance}$$

$$\text{C.V} = \text{S.D/mean}$$

ii) The distribution function $F_{MWW\ (\beta, \Upsilon, \theta, C)}$ of MWW (β, ϒ, θ, C) is given by:

$$F_{MWW(\beta,\gamma,\theta,c)}(X) = 1 - e^{-\beta(c\theta^{\gamma}+1)X^{\gamma}} \tag{4.2}$$

iii) The Reliability function $R_{MWW\ (\beta, \Upsilon, \theta, C)}$ of MWW (β, ϒ, θ, C) is given by:

$$\mathrm{R}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x}) = 1 - \mathrm{F}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})$$
$$= e^{-\beta(c\theta^{\gamma}+1)X^{\gamma}} \tag{4.3}$$

iv) The hazard function $h_{MWW\ (\beta, \Upsilon, \theta, C)}$ (X) of MWW (β, ϒ, θ, C) is given by:

$$\mathrm{h}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x}) = \frac{\mathrm{f}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x})}{\mathrm{R}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x})}$$
$$= \gamma\,\beta\,(c\,\theta^{\gamma}+1)\,X^{\gamma-1} \tag{4.4}$$

v) The reversed hazard function $r_{MWW\ (\beta, \Upsilon, \theta, C)}$ (X) of MWW (β, ϒ, θ, C) is given by:

$$\mathrm{r}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x}) = \frac{\mathrm{f}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x})}{\mathrm{F}_{\mathrm{MWW}}\ (\beta, \gamma, \theta, \mathrm{C})\ (\mathrm{x})}$$
$$= \frac{\gamma\,\beta\,(c\,\theta^{\gamma}+1)X^{\gamma-1}.e^{-\beta(c\theta^{\gamma}+1)X^{\gamma}}}{1-e^{-\beta(c\theta^{\gamma}+1)X^{\gamma}}} \tag{4.5}$$

vi) The mean reversed residual function m $_{MWW(\beta, \Upsilon, \theta, C)}$ of MWW (β, ϒ, θ, C) is given by:

$$m_{mww(\beta,\gamma,\theta,c)}(x) = \int_0^x F(t)dt / F(x)$$

$$= \left(1-e^{-\beta(c\theta^{\gamma}+1)X^{\gamma}}\right)^{-1} \sum_{i=1}^{\infty} (-1)^{i+1} (i!)^{-1} (i\gamma+1)^{-1} \quad (3.6)$$

5. Special Cases

(a) From eq. (2.5) the modified weighted probability density function

$y_1 = \log[\beta X^{\Upsilon}]$ is given by:

$$f(y_1) = e^{y_1} e^{-e^{y_1}}$$

$$f_{MWEVD}(y_1) = \left(c\theta^{\gamma}+1\right) e^{y_1} e^{-\left(C\theta^{\gamma}+1\right)e^{y_1}} \quad (5.1)$$

For $Y_1 > 0$ which is modified weighted extreme value distribution.

(b) The modified weighted probability density function of Y_2 for putting r = 2 in eq. (2.5) is given by:

For $Y_2 > 0$

$$f_{MWR}(y_2) = 2\beta(c\theta^2+1)x\, e^{-\beta(c\theta^2+1)x^2} \quad (5.2)$$

Eq. (5.2) is modified weighted Rayleigh distribution,

(c) The modified weighted probability density function of Y_3 for putting r = 1 in eq. (2.5) is given by:

For x > 0

$$f_{MWE}(y_3) = \beta(c\theta+1)e^{-\beta(c\theta+1)x} \quad (5.3)$$

Eq. (5.3) is Modified weighted Exponential distribution.

6. Parameter Estimation

From eq. (2.5) the log-likelihood function is given as:

Partially differentiating (6.1) w.r.t unknown parameters β, ϒ, θ and c we get:

$$\log L(x) = n\log\gamma + n\log\beta + n\log(c\theta^{\gamma}+1) + (\gamma-1)\sum_{i=1}^{n}\log x_i - (c\theta^{\gamma}+1)\beta\sum_{i=1}^{n}X_i^{\gamma} \quad (6.1)$$

$$\frac{\partial\log L(x)}{\partial\beta} = n/\beta - ((c\theta^{\gamma}+1)\beta\sum_{i=1}^{n}X_I^{r} \quad (6.2)$$

$$\frac{\partial\log L(x)}{\partial\gamma} = \frac{n}{\gamma} + \frac{n(c\theta^{\gamma}.\log\theta)}{(c\theta^{\gamma}+1)} + \sum_{i=1}^{n}\log x_i - \beta\,[c\theta^{\lambda}\log\theta\sum_{i=1}^{n}x_i^{\gamma} + (c\theta^{\gamma}+1)\sum_{i=1}^{n}x_i^{\gamma}\log(x_i) \quad (6.3)$$

$$\frac{\partial\log L(x)}{\partial\theta} = \frac{nc\gamma\theta^{\gamma-1}}{c\theta^{\gamma}+1} - \beta c\gamma\theta^{\gamma-1}\sum_{i=1}^{n}x_i^{\gamma} \quad (6.4)$$

$$\frac{\partial\log L(x)}{\partial c} = \frac{n\theta^{\gamma}}{c\theta^{\gamma}-1} - \beta\theta^{\gamma}\sum_{i=1}^{n}x_i^{\gamma} \quad (6.5)$$

We can solve the eq. (6.2) to (6.5) to get the MLEs of the unknown parameters λ= (β, ϒ, θ, c) simnltanionsly.

The information matrix is given as:

$$I(\lambda) = \begin{bmatrix} I_{11}(\lambda) & I_{12}(\lambda) & I_{13}(\lambda) & I_{14}(\lambda) \\ & I_{22}(\lambda) & I_{23}(\lambda) & I_{24}(\lambda) \\ & & I_{33}(\lambda) & I_{34}(\lambda) \\ & & & I_{44}(\lambda) \end{bmatrix}$$

Where

The value of $\left(\frac{\partial^2\log L(x)}{\partial\lambda_i\partial\lambda_j}\right)$ are given in Appendix A.

The variance and covariance matrix may be approximated as;

$$\underset{\sim}{\vee} = \begin{bmatrix} \vee_{11} & \vee_{12} & \vee_{13} & \vee_{14} \\ & \vee_{22} & \vee_{23} & \vee_{24} \\ & & \vee_{33} & \vee_{34} \\ & & & \vee_{44} \end{bmatrix} = I^{-1}(\lambda)$$

The asymptotic distribution of the MLE's of $(\hat{\beta},\hat{\gamma},\hat{\theta},\hat{c})$ is given as, see; Milla (1981)

$$\hat{\lambda}_i \sim \mathrm{N}(\lambda, \mathrm{V}) \quad (6.6)$$

Since "$\underset{\sim}{v}$" involve the corresponding MLE's in order to obtain an estimate of "$\underset{\sim}{v}$" which is given as;

$$\underset{\sim}{\vee} = I^{-1}(\hat{\lambda})$$

Then by using (6.6) approximate 100(1-α)% confidence intervels for the parameters.

$$\hat{\lambda}_i \pm Z_{\alpha/2}\sqrt{\hat{v}_i}$$

For i = 1, 2, 3, 4. Where $Z\alpha_{/2}$ is the α-th upper percentile of the standard normal distribution

7. Appendix: A

$$\frac{\partial^2 \log L(x)}{\partial\beta\,\partial\gamma} = -\left[c\theta^\gamma \log\theta \sum_{i=1}^{n} x_i^\gamma + (c\theta^\gamma + 1)\sum_{i=1}^{n}(x_i^\gamma \log x_i)\right] \tag{6.7}$$

$$\frac{\partial^2 \log L(x)}{\partial\beta\,\partial\theta} = -c\gamma\theta^{\gamma-1}\sum_{i=1}^{n} x_i^\gamma \tag{6.8}$$

$$\frac{\partial^2 \log L(x)}{\partial\beta\,\partial c} = -\theta^\gamma \sum_{i=1}^{n} x_i^\gamma \tag{6.9}$$

$$\frac{\partial^2 \log L(x)}{\partial\gamma\,\partial\theta} = \frac{nc(c\theta^\gamma + 1)[\gamma\theta^{\gamma-1}\log\theta + \theta^{\gamma-1}] - nc\gamma\theta^{2\gamma-1}\log\theta}{(c\theta^\gamma + 1)^2} - \beta c\left[\gamma\theta^{\gamma-1}\log\theta + \theta^{\gamma-1} + \gamma\theta^{\gamma-1}\sum_{i=1}^{n} x_i^\gamma \log x_i\right] \tag{6.10}$$

$$\frac{\partial^2 \log L(x)}{\partial\gamma\,\partial c} = \frac{n\theta}{(c\theta^\gamma + 1)^2} \tag{6.11}$$

$$\frac{\partial^2 \log L(x)}{\partial\theta\,\partial c} = n\gamma\theta^{\gamma-1}(c\theta^\gamma + 1)^{-1} - \beta\gamma\theta^{\gamma-1}\sum_{i=1}^{n} x_i^\gamma \tag{6.12}$$

$$\frac{\partial^2 \log L(x)}{\partial^2\beta} = -\frac{n}{\beta^2} \tag{6.13}$$

$$\frac{\partial^2 \log L(x)}{\partial^2 \gamma} = -\frac{n}{\gamma^2} + \frac{nc\log^2\theta.\theta^\gamma\left[c\theta^\gamma + 1 - c^2\theta^\gamma \log\theta\right]}{(c\theta^\gamma+1)^2}$$
$$-\beta c\log\theta\left[\theta^\gamma \log\theta\sum_{i=1}^{n} x_i^\gamma + \theta^\gamma \sum_{i=1}^{n}\left[x_i^\gamma \log x_i\right]\right]$$
$$-\beta\left[c\theta^\gamma \log\theta\sum_{i=1}^{n}(x_i^\gamma \log x_i) + (c\theta^\gamma+1)\sum_{i=1}^{n}\left[x_i^\gamma \log^2 x_i\right]\right] \qquad (6.14)$$

$$\frac{\partial^2 \log L(x)}{\partial^2 \theta} = \frac{nc\gamma(\gamma-1)\theta^{\gamma-2}(c\theta^\gamma+1) - nc^2\gamma^2\theta^{2\gamma-1}}{(c\theta^\gamma+1)^2} \qquad (6.15)$$

$$\frac{\partial^2 \log L(x)}{\partial^2 c} = \frac{-n\theta^{2\gamma}}{(c\theta^\gamma+1)^2} \qquad (6.16)$$

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