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On Projective Q-Curvature Inheritance in Projective Finsler Spaces

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Abstract

R. B. Misra [5] has discussed projectve transformation in Projective Finsler spaces. H. D. Pande et.al [6] have developed and studied projective curvature collineations in symmetric Finsler spaces. The concept of curvature inheritance in Finsler space was introduced by S. P. Singh [8]. The present author and S. P. Singh ([1], [2]) have further studied and developed curvature inheritance in recurrent Finsler spaces. The objective of the present paper is to study projective Q− curvature inheritance in projective Finsler spaces. Some special cases are also discussed at the end of the paper.

Keywords: Curvature inheritance, contra field, concurrent field, Lie-derivative

1 Introduction

We consider an n−dimensional Finsler F_n space equipped with a positively homogeneous fundamental function $F(x,\dot{x})$ of degree one in its direction argument $\dot{x}^i$. The fundamental metric tensors $g_{ij}(x,\dot{x})$ and $g^{ij}(x,\dot{x})$ and are symmetric in indices i and j and homogeneous of degree zero in $\dot{x}^i$. R. B. Misra [5] has defined the projective covariant derivative of a vector field $X^i(x,\dot{x})$ with the help of the projective connection parameters $\pi_{jk}^i(x,\dot{x})$ as follows

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_m X^i)\pi^m_{jk}\dot{x}^j + X^m \pi^i_{mk} \tag{1.1}$$

where $\pi_{jk}^i(x,\dot{x})$ is positively homogeneous function being defined by

$$\pi^i_{jk} = G^i_{jk} - \frac{1}{(n+1)}(2\delta^i_{(j}G^s_{k)s} + \dot{x}^i G^s_{skh}) \tag{1.2}$$

Such projective Finsler space is denoted by PF_n. Then the following identities hold good:

$$(a)\pi^i_{jks}\dot{x}^j = 0, (b)\dot{\partial}_j\pi^i_{ks}\dot{x}^j = 0, (c)\pi^i_{jk}\dot{x}^j = \pi^i_k \tag{1.3}$$

The commutation formulae involving the projective covariant derivative of a tensor $T_j^i(x,\dot{x})$ are expressed by

$$\dot{\partial}_h(T^i_{j((k))}) - (\dot{\partial}_h T^i_j)_{((k))} = T^r_j \pi^i_{rhk} - T^i_r \pi^r_{jhk} \tag{1.4}$$

and

$$2T^i_{j[((h))((k))]} = -\dot{\partial}_r T^i_j Q^r_{shk}\dot{x}^s + T^s_j Q^i_{shk} - T^i_s Q^s_{jhk} \tag{1.5}$$

where

$$Q^i_{jkh} = 2\partial_{[h}\pi^i_{k]j} - \pi^i_{rj[k}\pi^r_{h]} + \pi^r_{j[k}\pi^i_{h]r} \tag{1.6}$$

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is called the projective entity. The projective entities satisfy the following relation:

$$(a)\quad Q_k = -Q^i_{ki}, (b)\quad Q^i_k = Q^i_{hk}\dot{x}^h = Q^i_{rhk}\dot{x}^r\dot{x}^h, (c)\quad Q^i_{jki} = Q_{jk}. \tag{1.7}$$

We consider an infinitesimal point transformation [9]

$$\bar{x}^i = x^i + v^i(x)dt \tag{1.8}$$

where $v^i(x)$ is any vector field and dt means an infinitesimal constant. The Lie-derivative of a tensor field $T^i_j(x,\dot{x})$ and the connection parameters π_{jk}^i are given by

$$L_v T^i_j = T^i_{j((h))}v^h - T^h_j v^i_{((h))} + T^i_h v^h_{((j))} + (\dot{\partial}_h T^i_j)v^h_{((s))}\dot{x}^s \tag{1.9}$$

and

$$L_v \pi^i_j = v^i_{((j))((k))} + Q^i_{hjk}v^h + \pi^i_{jkh}v^h_{((s))}\dot{x}^s \tag{1.10}$$

respectively.

The commutation formulae involving the operators L_v and $((k))$ for the tensor $T^i_j(x.\dot{x})$ are given by

$$L_v(\dot{\partial}_l T^i_j) = \dot{\partial}_l(L_v T^i_j), \tag{1.11}$$

$$(L_v T^i_{j((k))}) - (L_v T^i_j)_{((k))} = T^h_j L_v \pi^i_{kh} - T^i_h L_v \pi^h_{kj} - (\dot{\partial}_h T^i_j)L_v \pi^h_{ks}\dot{x}^s \tag{1.12}$$

and

$$(L_v \pi^i_{jk})_{((l))} - (L_v \pi^i_{jl})_{((k))} = L_v Q^i_{hjk} + 2\dot{x}^s \pi^i_{rh[j} L_v \pi^r_{k]s}. \tag{1.13}$$

The infinitesimal point transformation (1.8) defines a projective motion if it transforms the system of geodesics into the same system. The necessary and sufficient condition for (1.8) to be a projective motion in PF_n is that the Liederivative of the connection coefficient π^i_{jk} with respect to (1.8) has the form

$$L\pi^i_{jk} = \delta^i_j \psi_k + \delta^i_k \psi_j \tag{1.14}$$

for certain non-zero covariant vector $\psi_j(x)$.

2 Projective Q- curvature inheritance in Finsler space PF_n

In this section we consider an infinitesimal transformation (1.8) which admits projective motion in projective Finsler space PF_n. We shall define and study the infinitesimal transformation which is a projecutive $Q-$curvature inheritance in the same space.

Definition 2.1. *In a projective Finsler space PF_n, if the projective entity Q^i_{jkh} satisfies the relation*

$$L_v Q^i_{jkh} = \alpha Q^i_{jkh} \tag{2.1}$$

with respect to vector $v^i(x)$, then the infinitesimal transformation (1.8) *is called $Q-$curvature inheritance. The entity $a(x)$ is a non-zero function.*

When the infinitesimal point transformation(1.8) defines a projective motion in the PF_n, the Lie-derivative of the connection cofficients π^i_{jk} satisfies the relation (1.14).

Using the condition (1.14) on equation (1.10) , we get

$$L_v Q^i_{jkh} = \delta^i_j \psi_{h((k))} - \delta^i_k \psi_{h((j))} + \delta^i_h \psi_{j((k))} - \delta^i_h \psi_{k((j))} \tag{2.2}$$

where we have used the facts

$$(a)\pi^i_{hjk}\dot{x}^h = 0, (b)\psi_s \dot{x}^s = 0. \tag{2.3}$$

Applying equation (2.1) in the above equation , it takes the form

$$\alpha(x) Q^i_{jkh} = 2\delta^i_{[j}\psi_{|h|((k))]} + 2\delta^i_h \psi_{[j((k))]} \tag{2.4}$$

where the index between two parallel bars is unaffected when we consider skew-symmetric part in [*jk*] .Accordingly we state

Theorem 2.1. *In a a projective Finsler space PF_n, which admits the projective $Q-$curvature inheritance , the projective entity Q^i_{jkh} is expressed in terms of the scalar function $\psi(x)$ in the form* (2.4).

When the projective Q -curvature inheritance becomes affine motion , the condition $L_v\pi_{jk}^i = 0$ is satisfied . In this case

$$\delta^i_j\psi_k + \delta^i_k\psi_j = 0. \tag{2.5}$$

Setting $i= j$ in the above equation, we obtain

$$(n + 1)\psi_k= 0$$

which implies

$$\psi_k= 0 \tag{2.6}$$

in view of (1.14)

Conversely, if (2.6) is true , then

$$L_v\pi^i_{jk}= 0 \tag{2.7}$$

in view of (1.14)

. It means $\psi_k= 0$ is the necessary and sufficient condition for the infinitesimal transformation (1.8) to be affine motion. In this case , the equation (2.4) takes the form

$$\alpha(x)Q^i_{jkh} = 0 \tag{2.8}$$

in view of (2.1).

This implies that the space is flat since $a(x)$ is non-zero scalar function. Hence we state

Theorem 2.2. *Every motion admitted in a projective Finsler spance PF_n is a projective $Q-$curvature inheritance if the space is flat .*

Applying the identity (1.12) on the projective entity Q^i_{jkh}, we have

$$\begin{aligned}L_v(Q^i_{jkh((l))}) - \alpha Q^i_{jkh((l))} = {} & [\delta^i_l\psi_r + \delta^i_r\psi_l]Q^r_{jkh} - [\delta^r_l\psi_j + \delta^r_j\psi_l]Q^i_{rkh} \\ & -[\delta^r_k\psi_l + \delta^r_l\psi_k]Q^i_{jrh} + [\delta^r_l\psi_h + \delta^r_h\psi_l]Q^i_{jkr} \\ & -[\delta^r_l\psi_s + \delta^r_s\psi_l](\dot\partial_r Q^i_{jkh})\dot x^s.\end{aligned} \tag{2.9}$$

in view of (1.12) and (2.1) provided the gradient vector $a_{((l))}$ is zero. Since the vanishing of the scalar function ψ_l is necessary and sufficient condition for the transformation (1.8) to be affine motion, the equation (2.9) reduces to

$$L_v(Q^i_{jkh((l))}) = \alpha Q^i_{jkh((l))}. \tag{2.10}$$

We thus state

Lemma 2.1. *When the projective $Q-$curvature inheritance admitted in a projective Finsler space PF_n becomes a motion, the covariant derivatives of the projective entity $Q^i_{jk\ h}$ satisfies the inheritance property* (2.10) *provided the gradient vector $a_{((l))}$ is zero.*

In a projective Finsler space PF_n, if the metric tensor satisfies the relation $L_v g_{ij}= 2Cg_{ij}$ for non-zero constant C, the infinitesimal transformation (1.8) is said to be homothetic transformation [4]. In the case that the projective Finsler space PF_n admits a homothetic transformation (1.8),the condition $L_v\pi^i_{jk} = 0$ holds .We have proved that the necessary and sufficient condition for $L_v\pi^i_{jk} = 0$

to be true is $\psi_k= 0$

Hence immediately from (1.13) we get

$$\alpha Q^i_{jkh} = 0. \tag{2.11}$$

If however the projective Finsler space PF_n admits projective $Q-$curvature inheritance, then in view of (2.1) , we get

$$\alpha Q^i_{jkh} = 0,$$

which implies that $Q^i_{jkh}= 0$. Consequently we state

Theorem 2.3. *Every homothetic transformation admitted in a projective Finsler space PF_n, is a projective $Q-$curvature inheritnce if the space is flat.*

In a projective Finsler space PF_n if the projective entity Q^i_{jkh} satisfies the relation

$$Q^i_{jkh((l))} = K_l Q^i_{jkh} \tag{2.12}$$

for a non-zero covariant vector K_l, then the space under consideration is called recurrent projective Finsler space and we denote it as RPF_n. The vector K_l is called projective recurrence vector [5].

Applying Lie-derivative operator to (2.12), we get

$$L_v Q^i_{jkh((l))} = (L_v K_l) Q^i_{jkh} + K_l \alpha Q^i_{jkh}. \tag{2.13}$$

Using (2.1),(2.12) and Lemma 1.1, the equation (2.13) yields

$$(L_v K_l) Q^i_{jkh} = 0,$$

which implies

$$Q^i_{jkh} = 0$$

since $L_v K_l$ is non-zero.

This contradicts the assumption that the RPF_n. is non-flat.. .Hence we state

Theorem 2.4. *A general recurrent projective Finsler space RPF_n, does not permit projective $Q-$curvature inheritnce if it becomes affine motion.*

3 Special cases

In this section we study and discuss two special cases of projective $Q-$curvature inheritance in PF_n and RPF_n.

3.1 Contra field $v^i_{((j))} = 0$, (3.1)

the vector field $v^i(x)$ determines a contra field.

We now consider a special projective $Q-$curvature inheritance in the form

$$\bar{x}^i = x^i + v^i(x)dt, v^i_{((j))} = 0. \tag{3.2}$$

In view of (1.14) and (3.2) , the equation (1.10) takes the form

$$Q^i_{jkh} v^h = \delta^i_j \psi_k + \delta^i_k \psi_j \tag{3.3}$$

Differentiating covariantly the above equation with respect to x^l and using equation (3.1) , we obtain

$$Q^i_{jkh((l))} v^h = \delta^i_j \psi_{k((l))} + \delta^i_k \psi_{j((l))} \tag{3.4}$$

which implies

$$2Q^i_{j[k|h|((l))]} v^h = \delta^i_j \psi_{[k((l))]} + \delta^i_{[k} \psi_{|j|((l))]} \tag{3.5}$$

In view of Theorem 2.1 , the above equation takes the form

$$2Q^i_{j[k|h|((l))]} v^h = \alpha Q^i_{jkh}. \tag{3.6}$$

Accordingly we state

Theorem 3.1. *In a recurrent projective Finsler space* RPF_n *which admits projective* $Q-$*curvature inheritance ,if the vector field* $v^i(x)$ *spans a contra field of the form* (3.2)*, the relation* (3.6) *holds good.*

In a recurrent projective Finsler space RPF_n the relation (3.6) takes the form

$$2Q^i_{j[k|h|}K_{((l))]}v^h = \alpha(x)Q^i_{jkl} \tag{3.7}$$

on assumption that the transformation (3.2) defines a projective $Q-$curvature inheritance in RPF_n also. Accordingly we state

Corollary 3.1. *In a recurrent projective Finsler* RPF_n*, which admits the projective* $Q-$*curvature inheritance of the form* (3.2) *, if the vector field* $v^i(x)$ *spans a contra field , the relation*

$$\alpha(x)Q^i_{jkl} = 2Q^i_{j[k|h|}K_{l]}v^h$$

is necessarily true.

3.2 Concurrent field

In a projective Finsler space PF_n, if the vector field $v^i(x)$ satisfies the relation

$$v^i_{((j))} = \lambda\delta^i_j \tag{3.8}$$

where λ is a non-zero constant, the vector field $v^i(x)$ determines a concurrent field.

In this case we shall consider a projective $Q-$curvature inheritance of the form

$$\bar{x}^i = x^i + v^i(x)dt, \quad v^i_{((j))} = \lambda\delta^i_j \tag{3.9}$$

Using (3.2),(1.14)and (3.7), we obtain the relation (3.3).

Taking covariant differentiation of relation (3.2) with respect to x^l and in view of (3.5) ,we obtain

$$Q^i_{jkh((l))}v^h + \lambda Q^i_{jkl} = \delta^i_j\psi_{k((l))} + \delta^i_k\psi_{j((l))} \tag{3.10}$$

which implies

$$Q^i_{j[k|h|((l))]}v^h + \lambda Q^i_{j[kl]} = \delta^i_j\psi_{[k((l))]} + \delta^i_{[k}\psi_{|j|((l))]}.$$

Now applying Theorem 3.1 on the above equation , it takes the form

$$2Q^i_{j[k|h|((l))]}v^h + 2\lambda Q^i_{j[kl]} = \alpha(x)Q^i_{jkl}, \tag{3.11}$$

Consequently we have

Theorem 3.2. *In a projective Finsler space* PF_n *which admits the projective* $Q-$*curvature inheritance of the form* (3.9)*, if the vector field* $v^i(x)$ *determines a concurrent field , the relation* (3.9) *holds good.*

Let us now assume that the space under consideration a recurrent one. In this case , the equation (2.8) takes the form

$$2Q^i_{j[|h|}K_{l]}v^h + 2\lambda Q^i_{j[kl]} = \alpha(x)Q^i_{klj}. \tag{3.12}$$

We have

Corollary 3.2. *In a recurrent projective Finsler space* RPF_n*, which admits the projective* $Q-$*curvature inheritance of the form* (3.9)*, if the vector field* $v^i(x)$ *determines a contra field ,the relation* (3.12)*is necessarily true.*

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