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On Pre-Hilbert Algebras Containing a nonzero Central Idempotent f such that $\|fa\|=\|a\|$ and $\|a^2\| \leq \|a\|^2$

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Abstract

Let A be a real pre-Hilbert algebra without divisors of zero, we prove that if A has dimension two and satisfying $\|a^2\| = \|a\|^2$, for all $a \in A$, Then A is isomorphic to a new classes of two dimensional pre-Hilbert algebras. We also characterize the pre-Hilbert algebraic algebras without divisors of zero and containing a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$, to be flexible algebras. Furthermore, we prove that if A contains a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all a in A, then the following statements are equivalent:

1. A is power commutative
2. A is third power associative
3. A is algebraic of degree two.

Key Words: Pre-Hilbert algebras, flexible, central idempotent, third power associative algebras.

1 – Introduction

Let A be a non-necessarily associative real algebra which is normed as real vector space. We say that a real algebra is a pre-Hilbert algebra, if it's norm $\|.\|$ come from an inner product $(.|.)$, and it's said to be absolute valued algebras, if it's norm satisfy the equality $\|ab\| = \|a\|\|b\|$, for all $a, b \in A$. We recall that the set of pre-Hilbert absolute valued algebras is contained in the set of pre-Hilbert algebras satisfying the identity $\|a^2\| = \|a\|^2$ for all $a \in A$. Note that, the norm of any absolute valued algebras containing a nonzero central idempotent (or finite dimensional) come from an inner product [2] and [3]. We assume that A is pre-Hilbert algebra, without divisors of zero and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. An interesting Rodriguez's theorem [6] assert that every two-dimensional real absolute valued algebra is isomorphic to $\mathbb{C}, \mathbb{C}^*, *\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (the real algebras obtained by endowing the space $\mathbb{C}$ with the product $x * y = \bar{x}y, x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively). We extend the above mentioned theorem to more general situation, indeed, we prove that if A has dimension two, then A is isomorphic to a new classes of two-dimensional pre-Hilbert algebras (section 3). Also we show, in section 4, that if A is algebraic algebra and contains a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all $a \in A$, then the following assertion are equivalent:

i) A is flexible.

ii) A has degree two and if $\{f, u, v\}$ is an orthogonal family, then $\{f, u, v, uv\}$ is too, where $u, v \in V := \{w \in A | (w|f) = 0\}$.

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And the counter example is given. Moreover, we prove that if A contains a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all $a \in A$, then the following statements are equivalent:
1. A is power commutative
2. A is third power associative
3. A is algebraic of degree two.

2 – Notation and preliminaries results

In this paper all the algebras are considered over the real numbers field $\mathbb{R}$.

Definition 2.1 Let B be an arbitrary algebra.
i - B is called flexible, if it's satisfy the identity $(x, y, x) = 0$ for all $x, y \in B$ (where $(.,.,.) = 0$ denote the associatore).
ii - We say that B is third power associative, if it's satisfy the identity $(x, x, x) = 0$ for all $x \in B$.
iii - B is said power commutative if any sub-algebras generated by a single element is commutative.
iv - B is called a division algebra if the operators L_x *and* R_x of left and right multiplication by x are bijective for all $x \in B \setminus \{0\}$.

v - An element a in B is said to be algebraic (of degree n) if the sub-algebra $B(a)$ generated by a is finite dimensional (of dimension n). We say that B is algebraic if all its elements are algebraic. B is said to be algebraic of bounded degree if there exist a non-negative integer number n such that $dim\, B \leq m$ for any element a in B. If this is the case, then the small such number m is called the degree of B. Clearly every finite-dimensional algebra is algebraic of bounded degree. The $(1, 2, 4, 8)$ theorem show that the degree of every finite-dimensional real division algebra is $1, 2, 4$ *or* 8.
We need the following relevant results:

Proposition 2.2 [1] If $\{x_i\}$ si a set of commuting element in a flexible algebra A over a field characteristic not two. Then the sub-algebra generated by the $\{x_i\}$ is commutative.
Theorem 2.3 [5]. Let A be a real commutative algebraic algebra without divisors of zero, then $dim\, A \leq 2$.
Lemma 2.4 [7]. Every algebra in which $x^2 = 0$ only if $x = 0$, contains a nonzero idempotent.

3 – Two dimensional pre-Hilbert algebras, satisfying $\|a^2\| = \|a\|^2$

Firstly, we would like to consider the general situation of a two-dimensional real algebra A. let $\{e_1, e_2\}$ be a basis of A and $\alpha, \beta, \lambda, \mu, \alpha', \beta', \lambda', \mu' \in \mathbb{R}$. The product in A is determined by the multiplication table

	e_1	e_2
e_1	$\alpha e_1 + \beta e_2$	$\lambda e_1 + \mu e_2$
e_2	$\alpha' e_1 + \beta' e_2$	$\lambda' e_1 + \mu' e_2$

(1)

Theorem 3.1. The algebra A determined by the table **(1)** is division algebra if and only if
1) $4(\alpha\mu - \beta\lambda)(\alpha'\mu' - \beta'\lambda') > (\alpha'\mu + \alpha\mu' - \beta'\lambda - \beta\lambda')^2$.
2) $4(\alpha\beta' - \beta\alpha')(\lambda\mu' - \mu\lambda') > (\alpha'\mu - \alpha\mu' - \beta'\lambda + \beta\lambda')^2$.

Proof. Let $\{e_1, e_2\}$ be a basis of A such that the multiplication of A is given by the table **(1)**. Then for an arbitrary element $a = xe_1 + ye_2$ in A, we have

$$L_a(e_1) = (x\alpha + y\alpha')e_1 + (x\beta + y\beta')e_2 \text{ and } L_a(e_2) = (x\lambda + y\lambda')e_1 + (x\mu + y\mu')e_2$$

So the matrix of L_a in the above basis can be expressed as follow

$$M_{L_a} = \begin{pmatrix} x\alpha + y\alpha' & x\lambda + y\lambda' \\ x\beta + y\beta' & x\mu + y\mu' \end{pmatrix}$$

We have

$$det(M_{L_a}) = x^2(\alpha\mu - \beta\lambda) + xy(\alpha'\mu + \alpha\mu' - \beta'\lambda - \beta\lambda') + y^2(\alpha'\mu' - \beta'\lambda')$$

So A is a division algebra if and only if $det(M_{L_a}) \neq 0$, which is equivalent to

$$4(\alpha\mu - \beta\lambda)(\alpha'\mu' - \beta'\lambda') > (\alpha'\mu + \alpha\mu' - \beta'\lambda - \beta\lambda')^2.$$

By the same way we have the right multiplication of a, R_a is invertible if and only if $det(M_{R_a}) \neq 0$, which is equivalent to $4(\alpha\beta' - \beta\alpha')(\lambda\mu' - \mu\lambda') > (\alpha'\mu - \alpha\mu' - \beta'\lambda + \beta\lambda')^2$. □

Let $A_1(\gamma,\delta), A_2(\gamma,\delta)$, and $A_3(\gamma,\delta)$, be the real pre-Hilbert algebras defined by the multiplication tables **(2)**, **(3)** and **(4)** respectively, with $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$ ($\mathbb{R}^*$: the set of nonzero real numbers). And let $\{e,u\}$ be an orthonormal basis (where e is a nonzero idempotent)

	e	u
e	e	$\gamma e + \delta u$
u	$-\gamma e - \delta u$	e

(2)
$A_1(\gamma,\delta)$

	e	u
e	e	$\gamma e + \delta u$
u	$-\gamma e + (2-\delta)u$	$-e$

(3)
$A_2(\gamma,\delta)$

	e	u
e	e	$\gamma e + \delta u$
u	$-\gamma e - (2+\delta)u$	$-e$

(4)
$A_3(\gamma,\delta)$

Remark 3.2

i) The real algebra given by table **(2)** is a division algebra for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$

ii) The real algebra given by table **(3)** is a division algebra, if and only if, $\gamma^2 + \delta^2 < 2\delta$

iii) The real algebra given by table **(4)** is a division algebra, if and only if $\gamma^2 + \delta^2 < -2\delta$

Proof. Consequence of the theorem 3.1 □

Lemma 3.3 The algebras $A_1(\gamma,\delta), A_2(\gamma,\delta)$, and $A_3(\gamma,\delta)$ satisfies the identity $\|a^2\| = \|a\|^2$ for all $a \in A$ and $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$.

Proof. According to remark 3.2, $A_1(\gamma,\delta)$, with $\gamma,\delta \in \mathbb{R}^*$, is a two-dimensional real division algebra. And let $a \in A_1(\gamma,\delta)$, can be written as $a = \varepsilon e + \zeta u$ (where $\{e,u\}$ is an orthonormal basis of $A_1(\gamma,\delta)$). So by a simple calculation we have $\|a^2\| = \|a\|^2$, similarly proof for the others cases $A_2(\gamma,\delta)$, and $A_3(\gamma,\delta)$. □

Lemma 3.4 Let A be a real pre-Hilbert algebra, without divisors of zero, and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then the following equalities hold for all orthogonal elements $x,y \in A$:

1) $(x^2|xy + yx) = 0$
2) $\|xy+yx\|^2 + 2(x^2|y^2) = 2\|x\|^2\|y\|^2$

Proof. The equality $\|x^2\|^2 = (\|x\|^2)^2$ gives meaning to a polynomial P with real coefficients of degree ≤ 3 in λ, identically null, such that:

$$P(\lambda) = 2(x^2|xy + yx)\lambda^3 + (\|xy+yx\|^2 + 2(x^2|y^2) - 2\|x\|^2\|y\|^2)\lambda^2 + 2(y^2|xy + yx)\lambda.$$

Thus

1) $(x^2|xy + yx) = 0$
2) $\|xy+yx\|^2 + 2(x^2|y^2) = 2\|x\|^2\|y\|^2$. □

Now we can state our main result in this section

Theorem 3.5 Let A be a two-dimensional real pre-Hilbert algebra, without divisors of zero, and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then, A is isomorphic to $A_1(\gamma,\delta), A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$.

Proof. According to lemma 2.4, A is a two-dimensional real division algebra, containing a nonzero idempotent e. And let $\{e,u\}$ be an orthonormal basis of A. Then there exists $\gamma,\gamma' \in \mathbb{R}$ and $\delta,\delta' \in \mathbb{R}^*$, such that

$$eu = \gamma e + \delta u \text{ and } ue = \gamma' e + \delta' u.$$

We have

$eu + ue = (\gamma + \gamma')e + (\delta + \delta')u$, which means by lemma 3.4 (1) That $\gamma = -\gamma'$ and $eu + ue = (\delta + \delta')u$. Since

$$0 = (u^2|eu + ue) = (\delta + \delta')(u^2|u),$$

Then $eu + ue = 0$ or $u^2 = \pm e$, we distinguish the following cases.

case 1: If $ue + eu = 0$, then by lemma 3.4 (2) we have $(e|u^2) = 1$, so $\|u^2 - e\|^2 = 2 - 2 = 0$. Consequently $u^2 = e$, thus A is isomorphic to $A_1(\gamma,\delta)$.

case 2: If $u^2 = -e$, then $(\delta + \delta')^2 = 4$ (lemma 3.4 (2)). That is $\delta + \delta' = 2$ *or* $\delta + \delta' = -2$

i) If $\delta + \delta' = 2$ then $\delta' = 2 - \delta$. So A is isomorphic to $A_2(\gamma,\delta)$.

ii) If $\delta + \delta' = -2$ then $\delta' = -2 - \delta$. So A is isomorphic to $A_3(\gamma,\delta)$. □

We get the following results.

Corollary 3.6 Let A be a two-dimensional real pre-Hilbert algebra, containing a nonzero central idempotent e, without divisors of zero and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then A is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.

Proof. According to theorem 3.5, the algebra A is isomorphic to $A_1(\gamma,\delta), A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$. Since e is a central idempotent, then we have the following cases:

i) If A is isomorphic to $A_1(\gamma,\delta)$, then $eu = -ue$ which is absurd.

ii) If A is isomorphic to $A_2(\gamma,\delta)$, then $\gamma = 0$ and $\delta = 1$. So A is isomorphic to $\mathbb{C}$

iii) If A is isomorphic to $A_3(\gamma,\delta)$, then $\gamma = 0$ and $\delta = -1$. So A is isomorphic to $\overset{*}{\mathbb{C}}$. □

Corollary 3.7 Let A be a two-dimensional real third power associative pre-Hilbert algebra, without divisors of zero and satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. Then A is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.

Proof. According to theorem 3.5, the algebra A is isomorphic to $A_1(\gamma,\delta), A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$ for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2\delta$. The identity $(u,u,u) = 0$ imply that $eu = ue$, so e is a nonzero central idempotent. We conclude that A is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (Corollary 3.6). □

Now we conclude the theorem of A. Rodriguez

Corollary 3.8 Let A be a two-dimensional real absolute valued algebra. Then A is isomorphic to $\mathbb{C}, \mathbb{C}^*, *\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.

Proof. Since A is a finite-dimensional real absolute valued algebra, then A satisfying $\|a^2\| = \|a\|^2$ for all $a \in A$. and it's norm comes from an inner product [2]. Using theorem 3.5, the algebra A is isomorphic to $A_1(\gamma,\delta), A_2(\gamma,\delta)$, or $A_3(\gamma,\delta)$for all $(\gamma,\delta) \in \mathbb{R}\times\mathbb{R}^*$, such that $\gamma^2 + \delta^2 < \pm 2$. We have

$$(ue|e) = \pm(ue|u^2) = \pm(u|e) = 0 \ \textit{and} \ (eu|e) = \pm(eu|u^2) = \pm(u|e) = 0$$

This imply that the two elements ue *and* u (respectively eu *and* u) are linearly dependent, thus $\gamma = 0$. Therefore

i) If A is isomorphic to $A_1(\gamma,\delta)$, then the identity $eu = -ue = \pm u$ imply that A is isomorphic to $\mathbb{C}^*$ or $*\mathbb{C}$

ii) If A is isomorphic to $A_2(\gamma,\delta)$, then the identity $\|eu\| = \|ue\| = \|u\|\|e\| = 1$, imply that, $\delta = 1$ which means that A is isomorphic to $\mathbb{C}$.

iii) If A is isomorphic to $A_3(\gamma,\delta)$, then the identity $\|eu\| = \|ue\| = \|u\|\|e\| = 1$, imply that, $\delta = -1$ which means that A is isomorphic to $\overset{*}{\mathbb{C}}$. □

4 - Pre-Hilbert algebras containing a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \le \|a\|^2$

We begin with the following preliminary results.

Proposition 4.1 Let A be real pre-Hilbert algebra containing a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \le \|a\|^2$ for all $a \in A$. Then the following equalities hold:

i) $\|a^2\| = \|a\|^2$

ii) $a^2 - 2(a|f)fa + \|a\|^2 = 0$

Proof. i) Let $a \in A$, having an orthogonal sum decomposition $\lambda f + u$, the equality $\|a^2\|^2 \le (\|a\|^2)^2$ can be written

$$\|\lambda^2 + 2\lambda f + u^2\|^2 \le (\lambda^2 + \|u\|^2)^2 \quad (1)$$

As $\|fx\| = \|x\|$, then $(fx|f) = (f|x)$ for all $x \in A$. The development of (1) gives

$$2\lambda^2\left((f|u^2) + \|u\|^2\right) + 4\lambda(fu|u^2) + \|u^2\|^2 - \|u\|^4 \le 0 \qquad ((f|u) = 0) \quad (2)$$

We replace λ by –λ we get

$$2\lambda^2\big((f|u^2) + \|u\|^2\big) - 4\lambda(fu|u^2) + \|u^2\|^2 - \|u\|^4 \leq 0 \quad (3)$$

We add (2) and (3), we have

$$2\lambda^2\,((f|u^2) + \|u\|^2) + \|u^2\|^2 - \|u\|^4 \leq 0 \quad (4)$$

Since (4) hold for all $\lambda \in \mathbb{R}$ and $\|u^2\|^2 - \|u\|^2 \leq 0$, then $(f|u^2) \leq -\|u\|^2$.
According to the Cauchy-Schwarz inequality, we have

$$|(f|u^2)| \leq \|f\|\|u^2\| \leq \|u\|^2.$$

Then $(f|u^2) = -\|u\|^2$ also $\|u^2\| = \|u\|^2$, thus

$$\begin{aligned}\|u^2 + \|u\|^2 f\|^2 &= \|u\|^2 + 2\|u\|^2(u^2|f) + \|u\|^2\|f\|^2\\ &= \|u\|^4 - 2\|u\|^4 + \|u\|^4\\ &= 0\end{aligned}$$

hence $u^2 = -\|u\|^2 f$. On the other hand $a^2 = \lambda^2 f + 2\lambda fu + u^2$, then

$$\begin{aligned}\|a^2\|^2 &= \|(\lambda^2 - \|u\|^2)f + 2\lambda fu\|^2\\ &= (\lambda^2 - \|u\|^2)^2 + 4\lambda^2\|u\|^2 \qquad ((fu|f) = (f|u) = 0)\\ &= (\lambda^2 + \|u\|^2)^2\\ &= \|a\|^4\end{aligned}$$

Therefore $\|a^2\| = \|a\|^2$

ii) We have

$$\begin{aligned}a^2 &= \lambda^2 f + 2\lambda fu + u^2\\ &= -\lambda^2 f + 2\lambda f(\lambda f + u) - \|u\|^2 f\\ &= 2\lambda f(\lambda f + u) - (\lambda^2 + \|u\|^2)f\\ &= 2(f|a)fa - \|a\|^2 f.\end{aligned}$$

□

Remark 4.2 Let $V = \{u \in A \mid (u|f) = 0\}$, then the following equalities hold:
i) $V = \{u \in A \mid u^2 = -\|u\|^2 f\}$
ii) $uv + vu = -2(u|v)f$ for all $u, v \in V$.
iii) The product $x \wedge y = xy - (xy|f)f$ for all $x, y \in V$ endows V of an anti-commutative algebra structure
Proof.
i) According to proposition 4.1 (i).
ii) Let $x, y \in V$, we have $x + y \in V$. Then $(x + y)^2 = -\|x + y\|^2 f$ thus $xy + yx = -2(x|y)f$.
iii) Let $x, y \in V$, we have

$$\begin{aligned}x \wedge y + y \wedge x &= xy - (xy|f)f + yx - (yx|f)f\\ &= xy + yx - (xy + yx|f)f\\ &= -2(x|y)f + 2(x|y)f\\ &= 0\end{aligned}$$

□

Lemma 4.3. Let A be a real pre-Hilbert algebraic algebra of degree two, without divisors of zero and containing a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all $a \in A$. Then $A\,(f, v) = A\,(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$, for every $v \in V$.

Proof. It suffices to prove that $fv = v$ or $fv = -v$ for every $v \in V$, without loss of generality we will assume that $v \neq 0$, according to remark 4.2 (i), $v^2 = -\|v\|^2 f$. since A is algebraic algebra of degree two and satisfying $\|a^2\| = \|a\|^2$for all $a \in A$ (proposition 4.1 (i)), then by corollary 3.6 A is power-commutative algebra. It is well known that $(aa^2)a = a(a^2a)$ for any element a in A, hence

$$v(v^2v) = -\|v\|^2 v(fv) \;and\; (v^2v)v = -\|v\|^2(fv)v.$$

So $v(fv) = (vf)v, as\ \|vf\| = \|fv\| = \|v\|\ and\ (fv|f) = (vf|f) = (v|f) = 0$, then $(vf)^2 = v^2 = -\|v\|^2 f.\ So\ fv = v\ or\ fv = -v$, thus $A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. □

In the next theorem we give some conditions implying that the real pre-Hilbert algebra to be flexible algebra.

Theorem 4.4. Let A be a real pre-Hilbert algebraic algebra, without divisors of zero and containing a nonzero central idempotent f such that $\|fa\| = \|a\|\ and\ \|a^2\| \leq \|a\|^2$for all $a \in A$. Then the following assertions are equivalent:

i) A is flexible;

ii) A has degree two and if $\{f,u,v\}$ is an orthogonal family, then $\{f,u,v,uv\}$ is too, where $u,v \in V$.
Proof. $i) \Rightarrow ii)$ Assume that A is flexible algebra, according to proposition 2.2 and theorem 2.3, A has degree two. Let $w := uv - (uv|f)f - (uv|u)u - (uv|v)v$, where $u,v \in V$, without loss of generality we assume that $w \neq 0$. We have

$$(w|f) = (w|u) = (w|v) = 0$$

Since A is flexible and $uv + vu = 0$ (remark 4.2 (ii)), then

$$0 = uw + wu = 2(uv|u)f + 2(uv|f)u$$

This gives us $(uv|u) = (uv|f) = 0$, similarly $(uv|v) = 0$. Thus $\{f,u,v,uv\}$ is an orthogonal family.
$ii) \Rightarrow i)$ Let $x,y \in V$ such that $\|x\| = \|y\| = 1$ and $z := y - (y|x)x$, without loss of generality we can assume that $z \neq 0$, wehave $(z|f) = (z|x) = 0$. Then

$$\begin{aligned} 0 &= (xz|x) \\ &= (x(y - (y|x)x)|x) \\ &= (xy + (y|x)f|x) \quad (x^2 = -f, (remark\ 4.2\ (i)) \\ &= (xy|x) \end{aligned}$$

The equality $\|fa\| = \|a\|$ imply that $(fa|f) = (f|a)$, for all $a \in A$. On the other hand

$$\begin{aligned} 0 &= (xz|f) \\ &= (x(y - (y|x)x)|f) \\ &= (xy + (y|x)f|f) \\ &= (xy|f) + (x|y) \end{aligned}$$

This means that $(xy|f) = -(x|y)$, in the same way, we get $(yx|f) = -(x|y)$. Moreover, we have

$$\begin{aligned} (xy)x - x(yx) &= (x \wedge y + (xy|f)f)x - x(y \wedge x + (yx|f)f) \\ &= (x \wedge y)x + x(x \wedge y) + ((xy|f)f - (yx|f)f)x \\ &= -2(x|x \wedge y)f + ((xy|f)f - (yx|f)f)x \quad (remark\ 4.2\ (ii)) \\ &= ((xy|f)f - (yx|f)f)x. \\ &= 0 \end{aligned}$$

Since A is algebraic algebra of degree two, then the sub-algebras $A(f,x)$ and $A(f,y)$ are of dimension two and isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (lemma 4.3). We have the following cases:
1) If f is the only idempotent of A, then $xf = fx = x$ for all $x \in A$. Which means that A is a unit algebra, hence for all $a = \lambda f + x \text{ and } b = \gamma f + y$ in A, we have

$$\begin{aligned} (ab)a - a(ba) &= [(\lambda f + x)(\gamma f + y)](\lambda f + x) - (\lambda f + x)[(\gamma f + y)(\lambda f + x)] \\ &= (xy)x - x(yx) \\ &= 0 \end{aligned}$$

Then A is flexible.
2) If f is not unique, then $xf = fx = -x$ for all $x \in V$. Otherwise, if there exist a nonzero element $y \in V$ such that $yf = fy = y$. Then

$$\begin{aligned} f(x + y) &= \pm(x + y) \qquad \text{(Lemma 4.3)} \\ -x + y &= \pm(x + y) \end{aligned}$$

This imply that $x = 0$ or $y = 0$, which is absurd. Therefore for all $a = \lambda f + x$ and $b = \gamma f + y$ in A, we have

$$\begin{aligned} (a,b,a) &= (\lambda f + x, \gamma f + y, \lambda f + x) \\ &= (\lambda f, \gamma f, x) + (\lambda f, y, x) + (x, \gamma f, \lambda f) + (x, y, \lambda f) + (x, y, x) \end{aligned}$$

Or $(x,y,x) = 0$ and $(\lambda f, \gamma f, x) + (x, \gamma f, \lambda f) = 0$. Then

$$\begin{aligned} (a,b,a) &= (\lambda f, y, x) + (x, y, \lambda f) \\ &= \lambda[(fy)x - f(yx) + (xy)f - x(yf)] \\ &= \lambda[-yx - f(y \wedge x + (xy|f)f) + (x \wedge y + (xy|f)f)f + xy] \\ &= \lambda[-yx + y \wedge x - x \wedge y + xy] \quad ((xy|f) = (yx|f) = -(x| = \lambda[(x|y)f - (x|y)f] \\ &= 0 \end{aligned}$$

Then A is flexible. □

Remark 4.5. In [4], we constructed an example of four-dimensional absolute valued algebra containing a nonzero central idempotent of degree four which is not flexible. The last imply that the condition algebraic algebra of degree two is necessary for A to be flexible.

In the rest of this section, we prove that, if A is a real pre-Hilbert algebra, without divisors of zero and contains a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all $a \in A$. Then the following statements are equivalent:

1. A is power commutative.
2. A is third power associative.
3. A is algebraic of degree two.

We need the following preliminary result.

Lemma 4.6. Let A be a real third power associative pre-Hilbert algebra, without divisors of zero and contains a nonzero central idempotent f such that $\|fa\| = \|a\|$ and $\|a^2\| \leq \|a\|^2$ for all $a \in A$, then $A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ for every $v \in V$.

Proof. It suffices to prove that $ev = v$ or $ev = -v$ for every $v \in V$, without loss of generality we will assume that $v \neq 0$. According to remark 4.2 (i), $v^2 = -\|v\|^2 f$ as A is a third-power associative algebra. Then a linearization of the identity $(x, x, x) = 0$ gives

$$[x^2, y] + [xy + yx, x] = 0 \quad (5)$$

where $[x, y]$ denotes the quantity $xy - yx$.

By putting $y = x^2$ in the equality (5), we get the well-known identity $(x, x^2, x) = 0$. So $(aa^2)a = a(a^2a)$ for any element a in A, hence

$$v(v^2v) = -\|v\|^2 v(fv) \; and \; (v^2v)v = -\|v\|^2 (fv)v$$

So $v(fv) = (vf)v = (fv)v$, on the other hand $\|vf\| = \|fv\| = \|v\|$ and $(fv|f) = (vf|f) = (v|f) = 0$. Then $(vf)^2 = v^2 = -\|v\|^2 f$. This imply $fv = v \; or \; fv = -v$, thus $A(v)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.

Note that in the general case, if the algebra A is power commutative. We have for every $a \in A$ the sub-algebra $A(a)$ is commutative, so $a^2a = aa^2$ which means that $(a, a, a) = 0$, then A is third power associative. □

In the next theorem we have the reciprocally case.

Theorem 4.7. Let A is a real pre-Hilbert algebra, without divisors of zero and containing a nonzero central idempotent f such that $\|fa\| = \|a\| \; and \; \|a^2\| \leq \|a\|^2$ for all $a \in A$. Then the following statements are equivalent:

1. A is power commutative;
2. A is third power associative;
3. A is algebraic of degree two.

Proof. $(1) \Rightarrow (2)$ By definition.

$(2) \Rightarrow (3)$ Assume that A is third power associative and let $v \in V$, we have $v^2 = -\|v\|^2 f$ (proposition 4.1 (i)). By lemma 4.6, $A(a)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$, and consequently A is algebraic of degree two.

$(3) \Rightarrow (1)$ Using lemma 4.3. □

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