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Some New Representations of the Binet's Function Involving Euler Sums

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Abstract

We consider a method for transforming divergent series arising from the Euler - Maclaurin formula into convergent ones. Applying it to the Stirling series for $\log\Gamma(z)$ we obtain some new representations of the J. Binet function. In the special case when the argument z is integer or half-integer number the formulas take an elegant form involving colored Euler sums. Note that the obtained equalities are still hypotheses since they are derived by formal manipulations on divergent double series. We verify the results numerically, which require a computation of the Euler and related sums with high precision. Some old and new algorithms for this purpose are commented.

Keywords: Euler-Maclaurin formula, Stirling's approximation, Binet's function, divergent series, Euler sums, series acceleration

2020 Mathematics Subject Classification: 40-08, 40A25, 65B15, 33E99.

1. Introduction and main results

Applying the Euler-Maclaurin summation formula to a certain sum S_n we arrive at a constant $C(S_n)$, which is not calculated automatically unlike the other terms in the asymptotic expansion of S_n. If the series S_∞ is convergent, of course $C(S_n)$ coincides with the sum of the series. The calculation of the constant $C(S_n)$ by the Euler-Maclaurin formula is possible with arbitrary precision, but there is still much to be desired in this method. Therefore, methods that lead to representations of this constant, other than the definition, are of interest.

Note that the Euler-Maclaurin formula, extended to infinity usually gives a divergent series for $C(S_n)$. In this regard, the first purpose of this study is to present two methods for converting this divergent series to a more definite expression. The first method gives an integral representation of this constant and the second a convergent series composed by simple computable recursive terms multiplied by Dirichlet series.
We illustrate the both methods by applying them to the Stirling series

$$\log\Gamma(a) \sim \left(a-\frac{1}{2}\right)\log(\mathrm{a}) - a + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)(2k-1)}\cdot\frac{1}{a^{2k-1}}. \qquad (1)$$

It is well known that (1) is an asymptotic expansion when $a\to\infty$, but we do not use the symbol ' ~ ' in this sense. Here a is a fixed complex number and (1) means that the Euler-Maclaurin formula (applied to $S_n = \sum_{j=0}^{n}\log(a+j)$) associates the constant $C(S_n)$ with the divergent series to the right.

Having (1) we do not need a formal definition of $C(S_n)$, but for completeness let us fix the following:
For a given $m\in\mathbb{Z}$ let $f\in C^\infty[m,\infty)$ and $S_n := \sum_{i=m}^{n} f(i)$. Recall the Euler-Maclaurin formula, a variant from [17]:

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$$S_n = \int_m^n f(x)\,dx + \frac{f(m)+f(n)}{2} + \sum_{k=1}^{\lfloor p/2\rfloor} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) + R_p\,,$$

where $\{B_i\}$ are the Bernoulli numbers and the remainder term R_p is expressible with the periodized Bernoulli functions $P_k(x) = B_k(x - \lfloor x\rfloor)$, namely $R_p = \frac{(-1)^{p+1}}{p!}\int_m^n f^{(p)}(x)P_p(x)dx$.

Now, letting formally $p \to \infty$ and remove the remainder term we arrive at the following. Assume that $f^{2k-1}(n), k = 1,2,3,\dots$ form a scale for $n \to \infty$ and there exists a constant C such that the asymptotic expansion

$$S_n \approx \int_m^n f(x)\,dx + C + \frac{f(n)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n)$$

holds true. Then, we can define the constant $C(S_n)$ by C. Moreover, we get the formal equality

$$C(S_n) = \frac{f(m)}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(m)\,. \qquad (2)$$

Note that the concept to define the asymptotic constant as the sum of a series $\sum_{i=1}^{\infty} f(i)$, which partialsumhasasymptotics by the Euler-Maclaurin formula is addressed to Ramanujan,see [11, Ch.XIII]. The constant $C(S_n)$ is equivalent to the *Euler-Maclaurin constant of* f ($C(f)$) introduced in this book. When $m = 1$ the two constants differ only in the choice of the origin (zero) a of the primitive of f used for determination of $C(f)$ and $C(S_n) = C(f) + \int_a^1 f(x)dx$

In particular, for $S_n = \sum_{j=0}^{n} \log(a+j)$ we obtain $C(S_n) = \frac{1}{2}\log a - \mu(a)$, where

$$\mu(a) = \log\Gamma(a) - \left(a - \frac{1}{2}\right)\log a + a - \frac{1}{2}\log(2\pi)$$

is the Binet's function.

Remark that the notation $J(a)$ is often used for this function, which is in honor of its founder Jacques Philippe Marie Binet. We avoided this because of its considerable use. With the notation $\mu(a)$ we follow [13], where the author carefully studied and generalized some results from the Binet's original "book-size treatise".

We are interested in applications of formula (2), or in the case, of the formal representation

$$\mu(a) \sim \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} \cdot \frac{1}{a^{2k-1}}\,.$$

The first transformation of the above divergent series uses the expansion $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n$ and the integral $n! = \int_0^{\infty} x^n e^{-x}dx$. It leads to the formula

$$\mu(a) = \int_0^{\infty} e^{-a\,x}\left[\frac{x}{e^x-1} - 1 + \frac{x}{2}\right]\frac{dx}{x^2}\,. \qquad (3)$$

We can write "=" in (3), since this relation is known, see [18, 3.6] (which refers to [19, 12.31]) or the equality before (13.13.6) in [11] for another proof. Actually, (3) is the Binet's first expression for $\log\Gamma(a)$.

The second transformation is more complicated. Starting with replacing B_{2k} by $\zeta(2k)$ in (1) we succeed to resumm the obtained double series into a series of certain integrals. Then, rearranging the integrals to such on intervals of equal length and expanding the obtained integrals around the both ends and the centers of the intervals, we found three new representations of $\log\Gamma(a)$. In particular, the second is

$$\mu(a) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{J_k(a)\sum_{n=2}^{\infty}\frac{S_{n-1}(a)}{n^k} - I_k(a)\sum_{n=2}^{\infty}\frac{C_{n-1}(a)}{n^k}\right\}, \qquad (4)$$

where $I_k(a) = \int_0^1 t^{k-1}\sin(2\pi a t)\,dt$, $\quad J_k(a) = \int_0^1 t^{k-1}\cos(2\pi a t)\,dt$

and $C_{n-1}(a)=\sum_{j=1}^{n-1}\frac{\cos(2\pi aj)}{n-j}, \quad S_{n-1}(a)=\sum_{j=1}^{n-1}\frac{\sin(2\pi aj)}{n-j}$.
Note that the magnitude of the k-th term of the series in (4) is $O(2^{-k})$.

Is the relation (4) an equality for $a \in (0,\infty)$? The way that (4) is obtained does not answer this question, since several times formal manipulations on divergent series are performed. Thus, our second goal in the paper is to verify numerically the new representations of the Binet's function. In fact,the correctness of (4) imply that of the other two – (8) and (9).

Next, if in (4) we set $a=n$, a positive integer, we obtain the representation

$$\mu(n) \sim \frac{1}{\pi}\sum_{k=2}^{\infty} Sh_k^{'} \cdot P_{k-1}\left(\frac{1}{2\pi n}\right) \tag{5}$$

where$\boldsymbol{Sh_k^{'}} \coloneqq \sum_{i=2}^{\infty}\frac{\boldsymbol{h_{i-1}}}{\boldsymbol{i^k}}$(with$\boldsymbol{h_i}=\sum_{j=1}^{i}\frac{\boldsymbol{1}}{\boldsymbol{j}}$denoting the *i*-th harmonic number) and the polynomials $\{P_l\}_{l=1}^{\infty}$ are defined by$\boldsymbol{P_{k-1}}\left(\frac{\boldsymbol{1}}{\boldsymbol{2\pi n}}\right)=-\boldsymbol{I_k(n)}$.

Note that the constants$\{Sh_k^{'}\}$ are in the class of Euler sums which are expressible by zeta values. Also, the polynomials$\{P_l\}$obey simple recursive relations, for example:

$$P_1(t)=t\,, \quad P_2(t)=t\,,$$
$$P_3(t)=t-3\cdot 2t^2P_1(t)=t-3^{(2)}t^3\,,$$
$$P_4(t)=t-4\cdot 3t^2P_2(t)=t-4^{(2)}t^3\,,$$
$$P_5(t)=t-5\cdot 4t^2P_3(t)=t-5^{(2)}t^3+5^{(4)}t^5\,.$$

The other five formulas for$\log\Gamma(a)$obtained by substituting $a=n$or$a=n-\frac{1}{2}$in (4) and its companions (8) and (9) are given in Section 3. They involve colored Euler sums (see e.g. [12]). We summarize the results in this direction in the following

Theorem 1.*Assume that the series and integral representation (7) of the Binet's function holds true for every* $a>0$. *Then the formulas (8),(4) and (9) take place for every* $a>0$. *In addition, when the argument* a *is an integer or half-integer number, the formulas (10),(5),(11) and (13-15) hold too.*

Explicit expressions for the general Euler sums (in terms of zeta values) are known roughly speaking in the half of the cases. However, for all constants appearing in the formulas, we adopt a computational approach. Some algorithms for calculation of Euler sums the reader can find in [1], [10] and [5, §7]. In the last article general multiple polylogarithmic values are considered. Fast algorithms for evaluation of Tornheim zeta function (a generalization of the Euler sums) can be found in [9] and [15].

Recently, a common approach to look for new identities about general Euler sums is by using of integer relation algorithms ([7]). A reliable check for a relation between the constants of a given set may need their values in thousands decimal digits.

That's how we get to the third line of our study, namely effective computation of Euler and related sums with high precision. Our specific task considered in Section 4 is to found with N digit precision the sequence of numbers $\{C_k\}_1^{cN}$, where C_k has the form $S_{1,k}^{\pm\pm}$ ([12]). For $\{S_{1,k}^{++}\}$ we use the explicit expression by zeta values and for $\{S_{1,k}^{--}\}$ we use the known connection with $\{S_{1,k}^{+-}\}$, as for the latter sequence we develop a new algorithm with complexity $O(N^2)$ multiplications.

We applied the above approach to obtain the constants $\{Sh_k^{'}\}_{k=2}^{345}$ $(Sh_k^{'}=S_{1,k}^{++}-\zeta(k+1))$ and $\{Sh_k^{=}\}_{k=1}^{345}$ $(Sh_k^{=}=\zeta(k+1)-S_{1,k}^{--})$ with 110 decimal digits. Then, using these calculations, we checked formula (4) for $a \in \{j\}_{j=1}^{30} \cup \{j-1/2\}_{j=1}^{30}$ with about 100 decimal digits. Details are described in Section 5.

In the same section, for the case when $2a-1$ is not an integer,we develop integral representations of the coefficients of $J_k(a)$ and $I_k(a)$ in (4) and verify the formula for $a \in \{\sqrt{2}j-1\}_{j=1}^{30}$.

2. Derivation of the formulas for the Binet's function

In order to obtain (3), by the Stirling's formula we write formally

$$\mu(a) \sim \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} \cdot \frac{1}{a^{2k-1}} = \sum_{k=1}^{\infty} \frac{B_{2k}\Gamma(2k-1)}{(2k)!\,a^{2k-1}} = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \int_0^{\infty} \left(\frac{x}{a}\right)^{2k-2} e^{-x} d\left(\frac{x}{a}\right)$$

$$\sim \int_0^{\infty} e^{-a\,x} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k-2}\,dx \sim \int_0^{\infty} e^{-a\,x} \left[\frac{x}{e^x-1} - 1 + \frac{x}{2}\right] \frac{dx}{x^2},$$

where we used the expansion of $\frac{x}{e^x-1}$ outside its radius of convergence.

To derive (4) we use the relation$B_{2k} = (-1)^{k-1}(2k)! \cdot \frac{2\zeta(2k)}{(2\pi)^{2k}}$. Then,

$$\mu(a) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k)! \cdot 2\zeta(2k)}{(2k)(2k-1)(2\pi)^{2k}} \cdot \frac{1}{a^{2k-1}} \sim 2a \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{(2\pi a n)^{2k-1}}.$$

Thus we arrived at the divergent series $F(t) = 1 - 2!\,t^2 + 4!\,t^4 - + \cdots$, which can be extracted from the even part of the classical series considered by Euler $f(x) = 1 - 1!\,x + 2!\,x^2 - 3!\,x^3 + - \cdots$. This series is summable (B^*), i.e by a variation of the Borel's integral method (see [11, §8.11]), to $f(x) = \frac{1}{x} e^{\frac{1}{x}} \int_0^x e^{-\frac{1}{u}} \frac{du}{u} = \int_0^{\infty} \frac{e^{-w}}{1+x\,w} dw$, provided x is not real and negative. The first equality implies

$$F(t) = \frac{1}{2}(f(it) + f(-it)) = \frac{1}{t}\left[cos\frac{1}{t}\int_0^1 sin\frac{1}{u}\frac{du}{u} - sin\frac{1}{t}\int_0^1 cos\frac{1}{u}\frac{du}{u}\right] =: \frac{1}{t}H(t).$$

With this notation, the double sum above takes the form

$$\mu(a) \sim \frac{1}{\pi}\left\{\frac{1}{1}H\left(\frac{1}{2\pi a}\right) + \frac{1}{2}H\left(\frac{1}{4\pi a}\right) + \frac{1}{3}H\left(\frac{1}{6\pi a}\right) + \cdots\right\}. \tag{6}$$

The change of the variable $v = 1/u$ leads to

$$H\left(\frac{1}{2n\pi a}\right) = \cos(2n\pi a)\int_{2n\pi a}^{\infty} \sin v \frac{dv}{v} - \sin(2n\pi a)\int_{2n\pi a}^{\infty} \cos v \frac{dv}{v},$$

and we see that every such integral includes the next. This suggest the regrouping

$$\int_{2\pi a}^{\infty} \cdot = \int_{2\pi a}^{4\pi a} \cdot + \int_{4\pi a}^{6\pi a} \cdot + \int_{6\pi a}^{8\pi a} \cdot + \cdots$$

$$\int_{4\pi a}^{\infty} \cdot = \int_{4\pi a}^{6\pi a} \cdot + \int_{6\pi a}^{8\pi a} \cdot + \int_{8\pi a}^{10\pi a} \cdot + \cdots$$

and so on. Then, expanding H terms in this way and substituting in (6) we get

$$\mu(a) \sim \frac{1}{\pi}\left\{\frac{\cos(2\pi a)}{1}\int_{2\pi a}^{4\pi a} \sin v \frac{dv}{v} - \frac{\sin(2\pi a)}{1}\int_{2\pi a}^{4\pi a} \cos v \frac{dv}{v}\right.$$
$$+ \left(\frac{\cos(2\pi a)}{1} + \frac{\cos(4\pi a)}{2}\right)\int_{4\pi a}^{6\pi a} \sin v \frac{dv}{v} - \left(\frac{\sin(2\pi a)}{1} + \frac{\sin(2\pi a)}{2}\right)\int_{4\pi a}^{6\pi a} \cos v \frac{dv}{v}$$
$$\left. + \cdots\right\}, \quad \text{i.e.}$$

$$\mu(a) \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{2\pi n a}^{2\pi(n+1)a} \sin v \frac{dv}{v} \sum_{j=1}^{n} \frac{\cos(2\pi j a)}{j} - \int_{2\pi n\,a}^{2\pi(n+1)a} \cos v \frac{dv}{v} \sum_{j=1}^{n} \frac{\sin(2\pi j a)}{j}\right].$$

With the next transformation we equal the integral supports

$$\mu(a) \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{2\pi n a}^{2\pi(n+1)a} \sum_{j=1}^{n} \frac{\sin(v - 2\pi j a)}{j}\frac{dv}{v}\right] \tag{7}$$
$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi a} \sum_{j=1}^{n} \frac{\sin(w + 2\pi(n-j)a)}{j}\frac{dw}{w + 2\pi n a}.$$

First we expand the functions under integral sign around the left end of the interval

$$\mu(a) \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_0^{2\pi a}\frac{\sin w\, dw}{w+2\pi na}\sum_{j=1}^{n}\frac{\cos(2\pi(n-j)a)}{j} + \int_0^{2\pi a}\frac{\cos w\, dw}{w+2\pi na}\sum_{j=1}^{n}\frac{\sin(2\pi(n-j)a)}{j}\right]$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi na}\int_0^{2\pi a}\sin w\sum_{k=0}^{\infty}\left(\frac{-w}{2\pi na}\right)^k dw\ \left(C_{n-1}(a)+\frac{1}{n}\right)\right.$$

$$\left. + \frac{1}{2\pi na}\int_0^{2\pi a}\cos w\sum_{k=0}^{\infty}\left(\frac{-w}{2\pi na}\right)^k dw\ S_{n-1}(a)\right] \Rightarrow$$

$$\mu(a) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\left[I_k(a)\sum_{n=1}^{\infty}\left(\frac{C_{n-1}(a)}{n^k}+\frac{1}{n^{k+1}}\right) + J_k(a)\sum_{n=1}^{\infty}\frac{S_{n-1}(a)}{n^k}\right], \qquad (8)$$

where for the last implication we change the variable $w = 2\pi at$ in the integrals and shifted the summation index k. Recall that $I_k(a)$ and $J_k(a)$ was defined with (4).

Now expand the integrand in (7), precisely the part $1/(w + 2\pi na)$, around the right end of the interval. Then, with$w = 2\pi a - x$we have

$$\mu(a) \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi a}\sum_{j=1}^{n}\frac{\sin(2\pi(n+1-j)a-x)}{j}\frac{dx}{2\pi(n+1)a-x}$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_0^{2\pi a}\frac{\cos x\, dx}{2\pi(n+1)a-x}\sum_{j=1}^{n}\frac{\sin(2\pi(n+1-j)a)}{j}\right.$$

$$\left. - \int_0^{2\pi a}\frac{\sin x\, dx}{2\pi(n+1)a-x}\sum_{j=1}^{n}\frac{\cos(2\pi(n+1-j)a)}{j}\right]$$

$$= \frac{1}{\pi}\sum_{n=2}^{\infty}\left[\frac{1}{2\pi na}\int_0^{2\pi a}\cos x\sum_{k=0}^{\infty}\left(\frac{x}{2\pi na}\right)^k dx\, S_{n-1}(a) - \frac{1}{2\pi na}\int_0^{2\pi a}\sin x\sum_{k=0}^{\infty}\left(\frac{x}{2\pi na}\right)^k dx\, C_{n-1}(a)\right]$$

$$\sim \frac{1}{\pi}\sum_{k=1}^{\infty}\left[J_k(a)\sum_{n=2}^{\infty}\frac{S_{n-1}(a)}{n^k} - I_k(a)\sum_{n=2}^{\infty}\frac{C_{n-1}(a)}{n^k}\right],$$

where we shifted appropriately the indices n and $k$using the change $x = 2\pi at$. Thus we obtained (4).

In a very similar way, expanding (7) around the center of the interval, we obtain

$$\mu(a) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\left[\bar{I}_k(a)\sum_{n=1}^{\infty}\frac{\bar{C}_{2n-1}(a)}{(2n+1)^k} + \bar{J}_k(a)\sum_{n=1}^{\infty}\frac{\bar{S}_{2n-1}(a)}{(2n+1)^k}\right], \qquad (9)$$

where $\bar{I}_k(a) = \int_{-1}^{1} t^{k-1}\sin(\pi at)\, dt$, $\quad \bar{J}_k(a) = \int_{-1}^{1} t^{k-1}\cos(\pi at)\, dt$

and $\bar{C}_{2n-1}(a) = \sum_{j=1}^{n}\frac{\cos(2n+1-2j)\pi a}{j}, \quad \bar{S}_{2n-1}(a) = \sum_{j=1}^{n}\frac{\sin(2n+1-2j)\pi a}{j}$.

Note that "the half" of the above integrals are zero, and the magnitude of the k-th term in (9) is $O(3^{-k})$.

3. Cases of integer and half-integer argument

Substituting the argument a in (8) with a positive integer n (and the summation index with i) we see that the all terms$\frac{S_{i-1}(a)}{i^k}$vanish, as well as the first summand, because of $I_1(n) = 0$. Thus we come to the formula

$$\mu(n) \sim \frac{1}{\pi}\sum_{k=2}^{\infty}(-1)^k Sh_k \cdot P_{k-1}\left(\frac{1}{2\pi n}\right), \qquad (10)$$

where $Sh_k := \sum_{i=1}^{\infty}\frac{h_i}{i^k}$and the polynomials $\{P_l(x)\}$ are defined in the introduction.

The same argument, applied to (4) gives the relation (5).

Now, if we substitute in (9) a with $n \epsilon \mathbb{N}$ and taking into account the obvious zeros and the symmetry, we obtain

$$\mu(n) \sim \frac{2(-1)^{n+1}}{\pi}\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}\frac{h_i}{(2i+1)^{2k}} \cdot \int_0^1 t^{2k-1}\sin(\pi n t)\,dt\right]. \quad (11)$$

The constants appearing in (11) are easily expressible by the colored Euler sums $S_{1,q}^{\pm\pm}$ (see [12, §7] for a definition). Indeed, if we denote in this manner $S_{1,q}^{e,o} := \sum_{odd\ i\geq 0}^{\infty}\left(\sum_{even\ j\geq 2}^{i}\frac{1}{j}\right)\frac{1}{i^q}$ and similarly $S_{1,q}^{o,o}$, $S_{1,q}^{o,e}$ and $S_{1,q}^{e,e}$, one easily can express the above four sums by these, and in reverse. In particular, $\sum_{i=1}^{\infty}\frac{h_i}{(2i+1)^{2k}} = 2S_{1,2k}^{e,o} = \frac{1}{2}\left(S_{1,2k}^{++} + S_{1,2k}^{+-} - S_{1,2k}^{-+} - S_{1,2k}^{--}\right)$.
Moreover, in [16] these constants are expressed by zeta values (see also [14]).

On the other hand, the integrals in (11) are expressible by the polynomials $\{P_l\}$. To see this, let us first write some more general expressions for the integrals $I_k(a)$ and $J_k(a)$. One easily can check the formulas

$$I_{k+1}(a) = \frac{k}{2\pi a}J_k(a) - \frac{\cos 2\pi a}{2\pi a}, \quad J_{k+1}(a) = \frac{\sin 2\pi a}{2\pi a} - \frac{k}{2\pi a}I_k(a), \quad k \geq 1, \quad (12)$$

which imply

$$I_{k+1}(a) = q_k\left(\frac{1}{2\pi a}\right)\frac{\sin 2\pi a}{2\pi a} - p_k\left(\frac{1}{2\pi a}\right)\frac{\cos 2\pi a}{2\pi a} + (-1)^{k/2} mod(k+1,2)\frac{k!}{(2\pi a)^{k+1}}$$

and

$$J_{k+1}(a) = p_k\left(\frac{1}{2\pi a}\right)\frac{\sin 2\pi a}{2\pi a} + q_k\left(\frac{1}{2\pi a}\right)\frac{\cos 2\pi a}{2\pi a} + (-1)^{(k+1)/2} mod(k,2)\frac{k!}{(2\pi a)^{k+1}},$$

where

$$p_k(x) = 1 - k^{(2)}x^2 + k^{(4)}x^4 - + \cdots \qquad \text{and} \qquad q_k(x) = kx - k^{(3)}x^3 + k^{(5)}x^5 - + \cdots .$$

Note that any sequence of polynomials $\{p_l(x)\}$, $\{q_l(x)\}$ or $\{P_l(x)\}$ easily express the other two. For example, with these notations we have

$$\int_0^1 t^{2k-1}\sin(\pi n t)\,dt = I_{2k}\left(\frac{n}{2}\right) = -p_{2k-1}\left(\frac{1}{\pi n}\right)\frac{\cos \pi n}{\pi n} = (-1)^{n+1}P_{2k-1}\left(\frac{1}{\pi n}\right).$$

Now, let us consider the three expressions (8), (4) and (9) when the argument is specified to $n - \frac{1}{2}$, $n \epsilon \mathbb{N}$. We denote $\boldsymbol{a_i} := \sum_{j=1}^{i}\frac{(-1)^{j-1}}{j}$ and $\boldsymbol{S_k^{=}} := \sum_{i=2}^{\infty}(-1)^i\frac{a_{i-1}}{i^k}$.
Clearly, $\sum_{i=1}^{\infty}(-1)^{i-1}\frac{a_i}{i^k} = S_{1,k}^{--}$ and $S_k^{=} = \zeta(k+1) - S_{1,k}^{--}$. Then, with these notations, in the above manner we obtain the three representations

$$\mu\left(n - \frac{1}{2}\right) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1} \cdot I_k\left(n - \frac{1}{2}\right), \quad (13)$$

$$\mu\left(n - \frac{1}{2}\right) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}S_k^{=} \cdot I_k\left(n - \frac{1}{2}\right), \quad (14)$$

$$\mu\left(n - \frac{1}{2}\right) \sim \frac{2}{\left(n - \frac{1}{2}\right)\pi^2}\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i-1}a_i}{(2i+1)^{2k-1}} \cdot p_{2k-2}\left(\frac{1}{\left(n - \frac{1}{2}\right)\pi}\right)\right]. \quad (15)$$

For the coefficients in (15) it is not seen a relation with other constants.

Remark 1. Since the integrals in these formulas are expressed by polynomials of $1/a$ ($a = n$ or $a = n - 1/2$), they can be considered as rearrangements of the Stirling series (1). Now, if we release $n$from the constraint of an integer, it turns out that we are back in the field of divergent series. On the other hand, convergent rearrangements of (1) are possible, as can be seen from the thorough paper [3]. However, the present formulas can provide some good approximations for $\mu(a)$. For example, using (11) we found

$$\mu(a) \approx \sum_{k=1}^{8} \frac{D_k}{10^k} P_{2k-1}\left(\frac{1}{\pi a}\right), \quad a \geq 2\,,$$

where$\{D_k\} = \{2.50400041, 1.0339124, 0.94827, 0.9972, 1.088, 1.2, 1.4, 1\}$, as the uniform error is less than 10^{-10}. For comparison, the similar approximation from (1) has such accuracy for $a \geq 3.5$.

4. Evaluation of the constants that appeared in the formulas

In what follows we will consider algorithms for calculation of some constants with a precision of N decimal places. However, to this end, the intermediate calculations often need to be performed with higher precision. So, we will say briefly that a calculation is performed with N digits precision, meaning actually $O(N)$ true decimal digits. With this convention it doesn't matter if we talk about decimal or binary digits. But, when we describe a concrete computation, we will be more precise.

For the verification of the true signs in a final result we will use the simple rule of coincidence of the signs in two calculations with parameters N and qN $(q > 1)$.

Starting with the constants in (5) let us recall the famous Euler's result (e.g. [12])

$$Sh_k = S_{1,k}^{++} = \frac{1}{2}\left((k+2)\zeta(k+1) - \sum_{j=2}^{k-1} \zeta(j)\zeta(k+1-j) \right), \quad k \geq 2\,. \tag{16}$$

Thus, the evaluation of the numbers $\{Sh_k\}_{k=2}^{n}$ defined to (10), or equivalently $Sh_k' = Sh_k - \zeta(k+1)$, $k = 2, \dots, n$, with Ndigits precision can be performed through the sequence of numbers $\{\zeta(j)\}_{j=2}^{n+1}$ obtained with the corresponding precision.

Effective algorithms for calculation of $\zeta(s)$ the reader can find in [4, 6, 9]. We say that a computational algorithm is *effective*, if it uses $O(N\,(\log N)^p)$ multiplications for N digits precision of the result. Also a computational scheme we call *good* if it needs $O\left(N^k\,(\log N)^p\right)$ multiplications for the same purpose. Thus, the mentioned algorithms are effective if applied to a single zeta value. However, we need $O(N)$ zeta values for a verification of (5), so the total calculation, based on this approach, is a good scheme.

In fact, for the calculation of the constants $\left\{Sh_k'\right\}_{k=2}^{c\,N}$ that appeared in (5) we used the Euler identity as the zeta values we take for granted from Wolfram Mathematica.

Let us point out some details for the reader who want to perform the calculations. When using long arithmetic in the computer algebra system Mathematica we prefer the operator SetPrecision[..]instead of N[..],just to know what happens.The computation N[Expr($C_1, C_2, \ldots$), prec], of an expression that involves only exact constants $C_1, C_2, \ldots$ can be modeled by SetPrecision[Expr(SetPrecision[C_1,prec'], ...), prec], but the error estimation is up to the user.

Note about a property of the system when working with floating point numbers. Then it decreases the initial precision according to some own assessment of the error, which usually is overestimated. So, to keep the aimed precision in a complicated calculation, the user has two options. The first is to start with appropriate larger precision and the second is to restore it permanently by the operator SetPrecision[..].

We must be especially careful with the operator Sum[..]. It works properly in the numerical context only when the summands are with the same signs and of relatively equal magnitude. It is better to use a cycle by restoring precision at every step.Also, although the setting of our numerical problem suggest that the operator SetAccuracy will fit better, our experiments show that with this operator the effect of ignoring signs is much stronger (at least for these calculations).

Our intended accuracy is 100 decimal digits. Then a verification of (5) needs about 340 summands from the series. We increase the aimed precision for the constants in the calculation to 110 digits and take the sum in (5) up to $k = 345$.

We obtained the approximation of $\{Sh_k'\}_{k=2}^{345}$ by (16) and SetPrecision[.,120] (verified by SetPrecision[.,200]). The results for $\{Sh_{5+10i}'\}_{i=0}^{34}$, are presented in Table 1, as in the brackets are shown the numbers of the digits after the decimal point. All digits are true after rounding, so it is possible that the last one or two digits to differ from these in the exact decimal representation of the constants.

Now, let us turn to the constants appearing in (14). The first two can be expressed exactly:$S_1^{=} = \frac{1}{2}(\zeta(2) - \log(2)^2)$ and $S_2^{=} = \frac{13}{8}\zeta(3) - \frac{3}{2}\zeta(2)\log(2)$. Next, given an integer N (intended accuracy), the calculation of the constants$S_k^{=} = \zeta(k+1) - S_{1,k}^{--},\ k = 1, \dots, cN$ can be reduced to that of $\{S_{1,k}^{+-}\}_{k=1}^{cN}$ via the identity (see [12])

$$S_{1,k}^{--} + (-1)^k S_{1,k}^{+-} = \bar{\zeta}(k)\log 2 - \sum_{j=1}^{k-1}(-1)^j\,\bar{\zeta}(j)\,\zeta(k+1-j)\,,$$

where $\bar{\zeta}(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$. The constants $S_{1,k}^{+-} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{h_n}{n^k}$ can be expressed by zeta values only for even k, but we calculate them in a unified way as follows. Introducing the notion$\boldsymbol{M_k(x) = \sum_{n=1}^{\infty}\frac{h_n}{n^k}x^n}$we have $S_{1,k}^{+-} = -M_k(-1)$. Then we can use a formula for the Euler transform of the function $M_k(x)$. Namely, it holds the following

Lemma 1. *Let*$x = \frac{-t}{1-t}$ *and* $t = \frac{-x}{1-x}$ *belong to* $[-1,1)$. *Then we have*

$$(i) Li_k(x) = -\sum_{n=1}^{\infty}\tilde{\sigma}_{k-1}\left(\frac{1}{1},\frac{1}{2},\dots,\frac{1}{n}\right)\frac{t^n}{n}\ ;$$

$$(ii) M_k(x) = Li_1(x)Li_k(x) - (1-t)\sum_{n=1}^{\infty}h_n\tilde{\sigma}_k\left(\frac{1}{1},\frac{1}{2},\dots,\frac{1}{n}\right)t^n,$$

where$\tilde{\sigma}_k(a_1,a_2,\dots,a_n) = \sum_{1\le i_1\le\cdots\le i_k\le n}a_{i_1}\cdots a_{i_k}$ *is the sum of all the* k*-tuples (as products) from the set* $\{a_1,\dots,a_n\}$ *allowing repetitions.*

Proof. The relation (i) is known, see e.g. [8] and the references therein. So, we need to prove only (ii).To this end we use the hypergeometric function, in particular the transformation formula ([2, §2.1,(22)]): $F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right)$ .We have

$$\begin{aligned}
A = \frac{1}{a}\frac{\partial}{\partial b}F(1,a;a+b;x)\Big|_{b=1} &= \frac{1}{a}\frac{\partial}{\partial b}\sum_{n=0}^{\infty}\frac{(a)_n}{(a+b)_n}x^n\Big|_{b=1}\\
&= \frac{1}{a}\sum_{n=1}^{\infty}\frac{(a)_n}{(a+1)_n}\left(-\sum_{j=1}^{n}\frac{1}{a+j}\right)x^n\\
&= -\sum_{n=1}^{\infty}\frac{x^n}{(a+n)^2} - \sum_{n=2}^{\infty}\frac{1}{a+n}\left(\sum_{j=1}^{n-1}\frac{1}{a+j}\right)x^n\\
&= -\sum_{n=1}^{\infty}\frac{x^n}{(a+n)^2} - \sum_{n=2}^{\infty}\sum_{j=1}^{n-1}\frac{1}{n-j}\left(\frac{1}{a+j}-\frac{1}{a+n}\right)x^n\\
&= -\sum_{n=1}^{\infty}\frac{x^n}{(a+n)^2} + \sum_{n=2}^{\infty}h_{n-1}\frac{x^n}{a+n} - \sum_{j=1}^{\infty}\frac{1}{a+j}\left(\frac{x^{j+1}}{1}+\frac{x^{j+2}}{2}+\frac{x^{j+3}}{3}+\cdots\right)\\
&= \sum_{n=2}^{\infty}h_{n-1}\frac{x^n}{a+n} - \sum_{n=1}^{\infty}\frac{x^n}{(a+n)^2} - Li_1(x)\sum_{j=1}^{\infty}\frac{x^j}{a+j} =: A_1
\end{aligned}$$

On the other hand

$$A=\frac{1}{a}\frac{\partial}{\partial b}\left((1-x)^{-1}F\left(1,b;a+b;\frac{x}{x-1}\right)\right)\Bigg|_{b=1}$$
$$=\frac{1}{a(1-x)}\frac{\partial}{\partial b}\sum_{n=0}^{\infty}\frac{(b)_n}{(a+b)_n}t^n\Bigg|_{b=1}$$
$$=\frac{1}{a(1-x)}\sum_{n=1}^{\infty}\frac{n!}{(a+1)_n}\left(h_n-\sum_{j=1}^{n}\frac{1}{a+j}\right)t^n$$
$$=\frac{1}{a(1-x)}\left(\sum_{n=1}^{\infty}\frac{n!}{(a+1)_n}h_nt^n+\frac{\partial}{\partial a}\sum_{n=1}^{\infty}\frac{n!}{(a+1)_n}t^n\right)=:A_2$$

Now, let us take the functional $\frac{\partial^{k-1}}{\partial a^{k-1}}\Big|_{a=0}$ at the both sides of the identity $A_1=A_2$, i.e. compare $(k-1)!$ times the coefficients of a^{k-1} in the Maclaurin series with respect to a. Then, using the known expansion $\frac{n!}{(a+1)_n}=\sum_{j=0}^{\infty}(-a)^j\tilde{\sigma}_j\left(\frac{1}{1},\dots,\frac{1}{n}\right)$, which is also easily seen by multiplying n series of the form $\frac{1}{a+a_i}=\frac{1}{a_i}\left(1-\frac{a}{a_i}+\frac{a^2}{a_i^2}-\cdots\right.$, we obtain

$$(-1)^{k-1}(k-1)!\left[\sum_{n=2}^{\infty}h_{n-1}\frac{x^n}{n^k}-k\sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}}-Li_1(x)Li_k(x)\right]=$$
$$=\frac{(k-1)!}{1-x}coef\Bigg|_{a^k}\left(\sum_{n=1}^{\infty}\left[t^n\left(h_n+\frac{\partial}{\partial a}\right)\sum_{j=0}^{\infty}(-a)^j\tilde{\sigma}_j\left(\frac{1}{1},\dots,\frac{1}{n}\right)\right]\right)$$
$$=\frac{(k-1)!}{1-x}\left((-1)^k\sum_{n=1}^{\infty}t^n\tilde{\sigma}_k\left(\frac{1}{1},\dots,\frac{1}{n}\right)h_n+(-1)^{k+1}(k+1)\sum_{n=1}^{\infty}t^n\tilde{\sigma}_{k+1}\left(\frac{1}{1},\dots,\frac{1}{n}\right)\right).$$

As a consequence, we get

$$\sum_{n=1}^{\infty}\frac{h_n}{n^k}x^n=(k+1)Li_{k+1}(x)+Li_1(x)Li_k(x)\qquad(17)$$
$$+\sum_{n=1}^{\infty}\left((k+1)\tilde{\sigma}_{k+1}\left(\frac{1}{1},\dots,\frac{1}{n}\right)-h_n\tilde{\sigma}_k\left(\frac{1}{1},\dots,\frac{1}{n}\right)\right)\frac{t^n}{1-x}.$$

Now we will use the simple recurrence formula

$$\tilde{\sigma}_{k+1}\left(\frac{1}{1},\frac{1}{2},\dots,\frac{1}{n}\right)=\tilde{\sigma}_{k+1}\left(\frac{1}{1},\dots,\frac{1}{n-1}\right)+\frac{1}{n}\tilde{\sigma}_k\left(\frac{1}{1},\dots,\frac{1}{n}\right),\qquad(18)$$

which holds for $n\geq 1$ if we adopt the convention $\tilde{\sigma}_k(\emptyset)=0$.
Then we have the alternative record of (i)

$$Li_k(x)=(t-1)\sum_{n=1}^{\infty}\tilde{\sigma}_k\left(\frac{1}{1},\frac{1}{2},\dots,\frac{1}{n}\right)t^n.$$

Adding to this formula the equality $(1-x)(1-t)=1$ we see a cancellation of the terms with $(k+1)$ in (17) and the lemma is proved. □

Using the lemma, one can evaluate a single constant $M_k(-1)$ with N digits by summing $O(N)$ terms of a series which converges like 2^{-n}. But, with increasing of k the terms in (ii) become more complicated. Then, for evaluation of the constants $\{M_k(-1)\}_{k=1}^{cN}$ with N digits precision, we can use (18), reducing the number of multiplications to $O(N^2)$. In addition, $O(N^2)$ more multiplications (of zeta values) are necessary to convert the sequence $\{M_k(-1)\}_{k=1}^{cN}$ to $\{S_k^{=}\}_{k=1}^{cN}$.

Applying such a computational scheme, we obtained the sequence $\{S_k^{=}\}_1^{345}$ with 110 decimal digits precision. The results are illustrated in Table 2.

Remark 2. We can write even faster converging series for the constants $\{\eta(k)\}$ and $\{S_{1,k}^{+-}\}$ but with not so explicit coefficients. Namely, if $x = \frac{2z}{1+z}$, so that $x = -1$ corresponds to $z = -\frac{1}{3}$, then we have

$$(i) Li_k(x) = \sum_{n=1}^{\infty} c_{k,n} z^n,$$

where$c_{1,2i-1} = \dfrac{2}{2i-1}$, $c_{1,2i} = 0$, i = 1,2,3,..., and

$c_{k+1,n} = \dfrac{C_{k,n}}{n}$, $C_{k,n} = c_{k,n} - C_{k,n-1}$ $(C_{k,0} = 0)$;

$$(ii) M_k(x) = \sum_{n=1}^{\infty} d_{k,n} z^n,$$

where$d_{0,2i} = 4\sum_{j=1}^{i} \dfrac{1}{2j-1}$, $d_{0,2i+1} = d_{0,2i} + \dfrac{2}{2i+1}$, i = 0,1,2,..., and

$d_{k+1,n} = \dfrac{D_{k,n}}{n}$, $D_{k,n} = d_{k,n} - D_{k,n-1}$ $(D_{k,0} = 0)$.

5. Verification of the new formulas for $\log \Gamma(a)$

First note that the series in the right hand side of (4), (8) and (9) are equal to the series of integrals in (7). So, it suffices one to prove the equality sign in (4) and then, as a consequence, (7-9) hold with equality as well.

Verification of (4) *for*$a \in \{j\}_{j=1}^{30}$.

We need to check (5) for $n \le 30$. In the previous section we described the computation of the constants Sh_k' for $k = 2, \ldots, 345$ with about 110 digits after the decimal point.Next, it turns out that the simpler calculation of the integrals $P_k\left(\frac{1}{2\pi n}\right) = -I_k(n)$ for $k = 2, \ldots, 345$ by the recurrence formulas is unstable. As a result, to obtain the aimed precision for these factors we started with precision of 500 digits. This gives the aimed precision for $n = 2, \ldots, 30$ and for$n = 1$ the absolute error of $\{2^{-k} I_k(n)\}_{k=2}^{345}$ is less than 10^{-110}, too.So, using these approximations we checked (5) for $n = 1,2, \ldots, 30$ observing coincidence of 107 decimal digits after the decimal point.

Verification of (4) *for*$a \in \left\{j - \frac{1}{2}\right\}_{j=1}^{30}$.

We need to check (14) as an equality for $n \le 30$. For the integrals appearing in (14) it holds a similar notion as for these in (5). Namely, their evaluation costs $O(N)$ multiplications, but the working precision has to be increased sensibly (5-6 times for $N = 100$). Using the already obtained values of $S_k^{=}$ for $k = 2, \ldots, 345$with 110 digits we confirmed (14) as an equality for $n = 1,2, \ldots, 30$ observing coincidence of 106 decimal digits between the left and the right hand side. We omit the details.

Verification of (4) *for*$a \in \left\{\sqrt{2}j - 1\right\}_{j=1}^{30}$.

Let us introduce the notations$\boldsymbol{\varphi_{\alpha,\beta}^{(k)}} \coloneqq \sum_{n=1}^{\infty} \frac{e^{i n\alpha}}{n^k} \boldsymbol{h_n^{\beta}}$, where$\boldsymbol{h_n^{\beta}} = \sum_{j=1}^{n} \frac{e^{i j\beta}}{j}$.

Clearly, with C_{n-1} and S_{n-1} defined at (4), we have

$$\varphi_{2\pi z,-2\pi z}^{(k)} = \zeta(k+1) + \sum_{n=2}^{\infty} \frac{C_{n-1}(z)}{n^k} + i \sum_{n=2}^{\infty} \frac{S_{n-1}(z)}{n^k} \tag{19}$$

Also, integrating by parts $\int_0^{\alpha} \varphi_{x,\beta}^{(k)}\, dx$ we obtain for $k \ge 1$ that

$$\sum_{j=0}^{k} \frac{(-i\alpha)^j}{j!} \varphi_{\alpha,\beta}^{(k+1-j)} = \sum_{n=1}^{\infty} \frac{h_n^{\beta}}{n^{k+1}} + \frac{(-i)^{k+1}}{k!} L_k(\alpha, \beta), \tag{20}$$

where $\boldsymbol{L_k(\alpha,\beta) = \int_0^\alpha x^k \frac{\log(1-e^{i(x+\beta)})}{1-e^{ix}} dx}$.

Note that the recurrence relation (20) reduces the calculation of $\left\{\sum_{n=2}^{\infty} \frac{C_{n-1}(\alpha)}{n^j}\right\}_{j=2}^{k}$ and $\left\{\sum_{n=2}^{\infty} \frac{S_{n-1}(\alpha)}{n^j}\right\}_{j=2}^{k}$ to that of the sequences $\left\{\varphi_{0,-\alpha}^{(j)}\right\}_{j=2}^{k}$ and $\left\{L_{j-1}(\alpha,-\alpha)\right\}_{j=2}^{k}$.

The initial value $\varphi_{\alpha,-\alpha}^{(1)}$ can be found in an explicit form. Namely, we have

Lemma 2. *For* $\alpha \in (0,2\pi)$ *it holds*

$$\varphi_{\alpha,-\alpha}^{(1)} = Li_1(e^{-i\alpha})\left[\frac{1}{2}Li_1(e^{-i\alpha}) + i\pi\right] + Li_2(e^{i\alpha}) - i\alpha Li_1(e^{i\alpha}) + \frac{\alpha^2}{2} - 2\zeta(2).$$

Sketch of the proof. By a formal series expansion we get

$$-\int_\varepsilon^\alpha \frac{\log(1-e^{i(x+\beta)})}{1-e^{ix}} dx \sim \frac{1}{i}\sum_{n=1}^{\infty} \frac{e^{in\alpha} - e^{in\varepsilon}}{n} h_n^{\beta}. \tag{21}$$

Denoting $f(\alpha) = -\int_\varepsilon^\alpha \frac{\log(1-e^{i(x-\alpha)})}{1-e^{ix}} dx$, with certain regularization for $x \to \alpha$, we obtain $f'(\alpha) = \frac{\log(1-e^{i\varepsilon}) - i\pi}{1-e^{i\alpha}} - \frac{\log[(1-e^{i(\alpha-\varepsilon)})(1-e^{i\alpha})]}{1-e^{i\alpha}}$, from where

$$f(\alpha) = -i\log\frac{1-e^{-i\alpha}}{1-e^{-i\varepsilon}}\log(1-e^{i\alpha}) - i\int_\varepsilon^\alpha \frac{y\,dy}{1-e^{iy}} + i\,\zeta(2) + o(1) \text{ for } \varepsilon \to +0.$$

On the other hand, for the sum $\varphi_{\varepsilon,\beta}^{(1)}$ that appears in (21), using summation by parts of its partial sums and the limit relation for $\varepsilon \to +0$ and $n \to \infty$

$$h_n^{\varepsilon} = h_n + O(\varepsilon) - \int_0^{n\varepsilon} \frac{1-e^{ix}}{x} dx = \log\frac{1}{\varepsilon} + i\frac{\pi}{2} + O\left(\varepsilon + \frac{1}{n\varepsilon}\right)$$

(with a constant independent of ε and n), we find

$$\varphi_{\varepsilon,\beta}^{(1)} = \left(\log\frac{1}{\varepsilon} + i\frac{\pi}{2} + O(\varepsilon)\right) Li_1(e^{i\beta}) - \sum_{n=2}^{\infty} h_{n-1}\frac{e^{in\beta}}{n}, \quad \varepsilon \to +0.$$

Now, in view of of the above limit expressions for $\varepsilon \to 0$, if we equalize the both sides of (21) with $\beta = -\alpha$ and represent $\int \frac{y\,dy}{1-e^{iy}}$ by Li_2, while $\sum h_{n-1}\frac{e^{in\beta}}{n}$ by Li_1^2, then we will obtain the assertion of the lemma with ~in place of =, i.e. as a hypothesis.

We did not investigate the area where (21) holds as an equality but again used numerical experiments to confirm the claimed equality for real α in $(0,2\pi)$. □

Lemma 3. *Let* $\alpha \in (0,2\pi)$ *and* N *be an integer number. Then the integrals* $L_k := L_k(\alpha,-\alpha)$, $k = 1,2,\dots,O(N)$, *can be calculated with* N *digits precision by* $O(N^2)$ *multiplications.*

Proof. The integrals have logarithmic singularity at $x = \alpha$. That is why we write

$$L_k = \int_0^\alpha \frac{x^k}{1-e^{ix}}\log(\alpha-x)\,dx + \int_0^\alpha \frac{x^k}{1-e^{ix}}\log\frac{1-e^{i(x-\alpha)}}{\alpha-x} dx \;=: \boldsymbol{L_k^{(1)}} + \boldsymbol{L_k^{(2)}}.$$

For $\left\{L_k^{(1)}\right\}$ let us expand $\frac{x}{1-e^{ix}}$ in Maclaurin series (involving Bernoulli numbers). Then, we need to calculate the values of the integrals $\boldsymbol{L_{k,j}^{(1)}} = \int_0^\alpha x^{k-1+j}\log(\alpha-x)\,dx$ for $j = 0,1,\dots,O(n)$. Indeed, the coefficients of the expansion decay like $(2\pi)^{-j}$, while (for a fixed k) $L_{k,j}^{(1)} = \frac{\alpha^{k+j}}{k+j}(\log\alpha - h_{k+j}) = o((2\pi)^j)$, and thus $L_k^{(1)} = \sum_j c_j L_{k,j}^{(1)}$ converges like a geometric series. We see that these calculations cost $O(N^2)$ operations.

For $\left\{L_k^{(2)}\right\}$ let us expand $\frac{x}{1-e^{ix}}\log\frac{1-e^{i(x-\alpha)}}{\alpha-x}$ in Taylor series around $x=\alpha/2$. This includes standard arithmetic operations on power series and one integration, so the expansion up to the $O(N)$-th term can be performed with $O(N^2)$multiplications (and even faster if one uses fast convolutions). In fact we used the operator Series[..] of Mathematica and will not go into details. Note that again this expansion serves for all $k\geq 1$.

Next, in view of the singularities of the function, the radius of convergence of the obtained series is $(2\pi-\alpha/2)$. On the other hand, the calculation involves the integrals $\boldsymbol{L_{k,j}^{(2)}}=\int_0^{\alpha}x^{k-1}\left(x-\frac{\alpha}{2}\right)^j dx=O\left((\alpha/2)^j\right)$ (k - fixed). Then, the quantities $L_k^{(2)}$ are represented by series converging like a geometric progression with ratio $q=\frac{\alpha}{4\pi-\alpha}<1$, and $O(N)$ terms are enough for N digits precision.

Finally, note that the integrals $\left\{L_k^{(2)}\right\}$ obey the simple recurrence relations

$$L_{k+1,j-1}^{(2)}=\frac{1}{j}\left(\alpha^k(\alpha/2)^j-kL_{k,j}^{(2)}\right)=\frac{\alpha}{2}L_{k,j-1}^{(2)}+L_{k,j}^{(2)},\quad k,j\geq 1\,,$$

which allow one to compute the necessary values for $k=1,\dots,O(N)$, $j=0,\dots,O(N)$ by $O(N^2)$ operations. (And the memory enough for the calculation is for $O(N)$long numbers.)□

Lemma 4.*Let*$\beta\,\epsilon\,\mathbb{R}\backslash 2\pi\mathbb{Z}$ *and* $N\,\epsilon\,\mathbb{N}$. *Then the quantities* $\varphi_{0,\beta}^{(k)}=\sum_{n=1}^{\infty}\frac{h_n^{\beta}}{n^k}$, $k=2,\dots,O(N)$ *can be calculated with* N *digits precision by* $O(N^2)$*multiplications.*

Proof. A summation by parts gives

$$\sum_{n=1}^{\infty}\frac{h_n^{\beta}}{n^k}=\zeta(k)Li_1\left(e^{i\beta}\right)-\sum_{n=2}^{\infty}\sum_{j=1}^{n-1}\frac{1}{j^k}\frac{e^{in\beta}}{n}=\zeta(k)Li_1(q)-\int_0^q\frac{Li_k(z)}{1-z}dz\,,$$

where $q=e^{i\beta}$. Note that, by the conditions of the lemma, $q\neq 1$.

Choosing an appropriate center c let us find the Taylor expansion of the function under the integral sign around $z=c$. It can be done easily recursively on k. Precisely, if $Li_k(z)=\sum_{n=0}^{\infty}d_{n,k}(z-c)^n$ and $\frac{Li_k(z)}{z}=\sum_{n=0}^{\infty}e_{n,k}(z-c)^n$, then we have

$$e_{n,k}=\frac{1}{c}\left(d_{n,k}-e_{n-1,k}\right),\quad e_{-1,k}:=0\,;$$
$$d_{n,k+1}=\frac{1}{n}\left(e_{n-1,k}\right),\quad d_{0,k+1}=Li_{k+1}(c)\,,$$

as the first relation is just by removing the denominator $z=c+(z-c)$, while the second is from$Li_{k+1}'(z)=\frac{1}{z}Li_k(z)$.

In this way, we can find every next series expansion of $Li_{k+1}(z)$ up to $(x-c)^{O(N)}$ by $O(N)$ multiplications. Another $O(N)$ operations are enough to expand$Li_{k+1}(z)/(1-z)$ up to the $O(N)$-th term. Summarily this amounts to $O(N^2)$ multiplications for finding the all partial sums of the series.

It remains to show that the integrals in $[0,q]$ of these power series give geometrically convergent number series that can be truncated at$O(N)$-th term. The radius of convergence of the all series is $|1-c|$. Then, the choice $c=q/2$ will give a convergence radius bigger than $1/2$ (since $c\,\epsilon\,\{|z|=1/2\}$ but $c\neq 1/2$, which is the closest point of the circle to $z=1$). On the other hand, the coefficients are combined with $\int_0^q(z-c)^n dz=O(2^{-n})$ and indeed this choice is appropriate.□

Corollary 1. Given $\alpha=2\pi a\,\epsilon\,\mathbb{R}$ and N -an integer,the values of $\varphi_{\alpha,-\alpha}^{(k)}$, $k=2,\dots,O(N)$,i.e.of $\left\{\sum_{n=2}^{\infty}\frac{C_{n-1}(a)+iS_{n-1}(a)}{n^k}\right\}_{k=2}^{O(N)}$,canbecalculated with$N$digits precision by$O(N^2)$multiplications.

Proof. It is clear that the reduction of α through 2π is admissible, so we can suppose that $\alpha\,\epsilon\,[0,2\pi)$. Now if $\alpha=0$ we obtain the particular case considered in the previous section. If $\alpha\,\epsilon\,(0,2\pi)$ the assertion follows from the recurrence relations (20) and Lemmas 2-4.□

Remark also the simple relation allowing us to reduce $\alpha \epsilon [0,\pi]$:

$$\overline{\varphi_{\alpha,-\alpha}^{(k)}} = \varphi_{-\alpha,\alpha}^{(k)}, \quad \alpha \epsilon \mathbb{R} \tag{22}$$

and which can accelerate the calculation of the integrals L_k.

Let us sketch the check of (4) for $a = \sqrt{2} - 1$. We choose to compare $N = 100$ decimal digits of the both sides of (4). To this end we take the sum on k up to $m = 345$ and distinguish the following steps:

- The integrals $I_k(a)$ and $J_k(a)$ for $k = 1, \dots, m$ we calculate by the recursive formulas (12). Because of instability we used precision 800 decimal digits and observe that the exactness of $I_k(J_k)$ gradually decreases with k from 800 to around 230 true digits. Finally we cut all these numbers at the 110-th digit.
- The calculation of the integrals $\{L_k\}_{k=1}^{m-1}$ follows the proof of Lemma 3. The power series needed for $\left\{L_k^{(1)}\right\}_{k=1}^{m-1}$ we expand up to$m_1 = 300$-th term exactly, while that for $\left\{L_k^{(2)}\right\}_{k=1}^{m-1}$ we expand up to$m_2 = 200$ -th term with precision$prec_2 = 150$ digits (introduced by making the parameter $\alpha = 2\pi\{a\}$ a floating point number). The number series induced from integrating the power series are rounded to 120 decimal digits.
- The evaluation of the quantities $\left\{\varphi_{0,\beta}^{(k)}\right\}_{k=2}^{m}$ follows the proof of Lemma 4. The series involved are expanded up to the $m_3 = m$ -th term, as first we make the parameter $q = e^{i\beta}$ a floating point number with $prec_3 = 120$ decimal digits. We updated this precision several times during the calculation.
- The numbers $\left\{\varphi_{\alpha,-\alpha}^{(k)}\right\}_{k=1}^{m}$ are computed by the recurrence formula (20) while at the same time they are rounded to precision 110 decimal digits.
- The sum in (4) is calculated up to $k = m$ and compared with the Binet's function. To avoid a fall in precision, we used the operator Do[..] instead of Sum[..]. Finally, we observed that the difference between the left and the right hand sides of (4) in this case is less than 10^{-108}.

Remark 3. Let us point out two problems that we had to solve at the last point.

First we expressed the coefficients $A_k = \sum_{n=2}^{\infty} \frac{C_{n-1}(a)}{n^k}$ and $B_k = \sum_{n=2}^{\infty} \frac{S_{n-1}(a)}{n^k}$ by $\varphi_{\alpha,-\alpha}^{(k)}$ (see (19)) directly in the sum on k. However it turns out that the imaginary parts of the complex numbers $\left\{\varphi_{\alpha,-\alpha}^{(k)}\right\}$, which were calculated with precision 110 digits, have significantly smaller precision for large k (since they are much smaller than the real parts). So, we had to extract A_k and B_k separately and to reset the precision before the summation.

Another unexpected fall in precision we encountered in the calculation $A_k = Re\left[\varphi_{\alpha,-\alpha}^{(k)}\right] - \zeta(k+1)$. We relied on that in the context of 110 digits precision the value of$\zeta(k+1)$ also will be computed with about 110 digits. Indeed, this is the case when $k+1$ is odd. However, when $k+1$ is even and k - large, the zeta value appeared with much smaller precision, as well as the final result. For this case we settled on the solution: prec=110; A_k=SetPrecision[$Re\left[\varphi_{\alpha,-\alpha}^{(k)}\right]$ − N[$\zeta(k+1)$,prec], prec].

For the readers who want to perform the calculations, we provide for comparison some values of the coefficients A_k and B_k in Table 3.

For the check of (4) with $a = j\sqrt{2} - 1$ let us introduce $\Delta_j \coloneqq LHS(4) - RHS(4)$. Then, similarly as above, but with parameters $m_1 = 300$, $m_2 = 200$, $prec_2 = 210$ and $m_3 = 400$, and using (22) if necessary, we verified that $|\Delta_j| < 10^{-101}$ for $j \epsilon \{2, \dots, 30\} \backslash M$, where $M = \{5,6,7,10,12,17,19,22,24,29\}$ and $|\Delta_6| < 10^{-98}$. Each of these checks lasted about 150 s on an ordinary machine.

Next, with the same parameters, but $m_3 = 800$ (and $prec_2 = 220$ for $j = 10$), we verified that $|\Delta_j| < 10^{-107}$ for $j = 10, 19, 22$ and $|\Delta_7| < 10^{-102}$. Also, with $m_3 = 2000$ we obtain $|\Delta_5|, |\Delta_{24}| < 10^{-107}$ and $|\Delta_{17}| < 10^{-59}$, while $m_3 = 4000$ gives $|\Delta_{12}| < 10^{-60}$. Finally, with $m = 175$, $m_3 = 20000$ and $prec_3 = 80$ we get $|\Delta_{29}| < 10^{-55}$, as the last check took 28 min.

Table 1. Coefficients of the formula (5), rounded to the (Nth) decimal sign.

k	$Sh_k^{'}$	k	$Sh_k^{'}$
5	0.04053689...77434322(109)	185	0.00000000...914160464(110)
15	0.00003062...95664423(109)	195	0.00000000...756875286(110)
25	0.00000002...25569760(109)	205	0.00000000...442660503(110)
35	0.00000000...38699109(109)	215	0.00000000...883550122(110)
45	0.00000000...58728184(109)	225	0.00000000...756684449(110)
55	0.00000000...262641463(110)	235	0.00000000...558356137(110)
65	0.00000000...243454175(110)	245	0.00000000...501521832(110)
75	0.00000000...289615092(110)	255	0.00000000...372560080(110)
85	0.00000000...866898770(110)	265	0.00000000...771848203(110)
95	0.00000000...563179927(110)	275	0.00000000...804464696(110)
105	0.00000000...694477450(110)	285	0.00000000...036918423(110)
115	0.00000000...168893394(110)	295	0.00000000...272496991(110)
125	0.00000000...228312514(110)	305	0.00000000...905539548(110)
135	0.00000000...608212728(110)	315	0.00000000...433501503(110)
145	0.00000000...661595547(110)	325	0.00000000...023860841(110)
155	0.00000000...447036104(110)	335	0.00000000...428734239(110)
165	0.00000000...582459580(110)	345	0.00000000...001395248(110)
175	0.00000000...078821696(110)		

Table 2. Coefficients of the formula (14), rounded to the (Nth) decimal sign.

k	$S_k^{=}$	k	$S_k^{=}$
5	0.029901635...82486286(109)	185	0.00000000...890682575(110)
15	0.000030483...77811113(109)	195	0.00000000...006362604(110)
25	0.000000029...42357991(109)	205	0.00000000...790850135(110)
35	0.000000000...88120060(109)	215	0.00000000...831074183(110)
45	0.000000000...83904225(109)	225	0.00000000...756683562(110)
55	0.000000000...099061802(110)	235	0.00000000...558356138(110)
65	0.000000000...656611598(110)	245	0.00000000...501521834(110)
75	0.000000000...758577791(110)	255	0.00000000...372560082(110)
85	0.000000000...082813126(110)	265	0.00000000...771848205(110)
95	0.000000000...653453795(110)	275	0.00000000...804464697(110)
105	0.000000000...577450460(110)	285	0.00000000...036918424(110)
115	0.000000000...540168353(110)	295	0.00000000...272496992(110)
125	0.000000000...548264596(110)	305	0.00000000...905539549(110)
135	0.000000000...805155692(110)	315	0.00000000...433501505(110)
145	0.000000000...725669674(110)	325	0.00000000...023860843(110)
155	0.000000000...160060835(110)	335	0.00000000...428734241(110)

165	0.000000000...122944371(110)	345	0.00000000...001395250(110)
175	0.000000000...732849460(110)		

Table 3. Coefficients to $I_k(a)$and$J_k(a)$ in formula (4) for $a = \sqrt{2} - 1$.

k	$-A_k(a)$	$B_k(a)$
5	0.02679148...66547461(109)	0.01405728...60347350(109)
15	0.00002618...25613628(109)	0.00001562...59930575(109)
25	0.00000002...21658852(109)	0.00000001...58689605(109)
45	0.00000000...83292562(109)	0.00000000...703494287(110)
65	0.00000000...670311529(110)	0.00000000...864466641(110)
85	0.00000000...370496073(110)	0.00000000...755404021(110)
105	0.00000000...633729965(110)	0.00000000...762870820(110)
125	0.00000000...683326225(110)	0.00000000...993276954(110)
145	0.00000000...858667117(110)	0.00000000...759693562(110)
165	0.00000000...020090985(110)	0.00000000...797375261(110)
185	0.00000000...095683026(110)	0.00000000...039906345(110)
205	0.00000000...922868855(110)	0.00000000...358613184(110)
225	0.00000000...238557829(110)	0.00000000...845282971(110)
245	0.00000000...852504004(110)	0.00000000...363817067(110)
265	0.00000000...168858387(110)	0.00000000...327664722(110)
285	0.00000000...481962365(110)	0.00000000...143269851(110)
305	0.00000000...024854166(110)	0.00000000...842773574(110)
325	0.00000000...590757394(110)	0.00000000...953172534(110)
345	0.00000000...001197425(110)	0.00000000...000716165(110)

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